

BROWNIAN MOTION AND STOCHASTIC FLOW SYSTEMS

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Introduction

From the standpoint of applications, Brownian motion may be the most important single topic in the theory of stochastic processes. This book provides a systematic exposition of the subject, emphasizing material of greatest interest in engineering, economics, and operations research. It is intended for researchers and advanced graduate students in those fields. About two-thirds of the book is devoted to development of the mathematical methods needed to analyze processes related to Brownian motion. The other third describes an area of application that I call stochastic flow systems, or the theory of buffered flow. Along the way, most of the important formulas related to Brownian motion are derived. As mathematical prerequisites, readers are assumed to have knowledge of

- elementary real analysis, including Riemann–Stieltjes integration, at the level of Bartle (1976),
- general measure and integration theory at the level of Royden (1968), and
- measure theoretic probability, including conditional expectation, at the level of Chung (1974).

In addition, a knowledge of elementary stochastic processes and some previous exposure to Brownian motion would be helpful. I recommend Çinlar (1975) for the former and Breiman (1968) for the latter.

Although it is aimed at readers with a strong mathematical background, I have tried to make the book accessible to others who may lack some of the prerequisite knowledge assumed above. Certain essential results from probability theory and real analysis are collected in the appendices, many important definitions are reviewed in the text, and correlative references are often given. With this help, I believe that mathematically able readers who have at least a nodding acquaintance with σ -algebras will be able to get by. Also, the many formulas should make this a useful reference for readers interested in specific applications as opposed to mathematical methods or foundations. In

summary, I hope this book will be immediately useful to readers with limited mathematical background, and may also serve to stimulate and guide further study.

A substantial portion of this book is devoted to a process generally known as "reflected Brownian motion," which is here called by another name. There is a prejudice among scholars against the coining of new terminology, but I feel that the old name is a major impediment to understanding. For an explanation of the problem, let $X = \{X_t, t \geq 0\}$ be a standard Brownian motion (zero drift and unit variance, starting at the origin) and then define

$$(1) \quad Z_t \equiv X_t - \inf_{0 \leq s \leq t} X_s, \quad t \geq 0.$$

Many years ago it was shown by P. Levy that this new process Z has the same distribution as Y , where

$$(2) \quad Y_t \equiv |X_t|, \quad t \geq 0.$$

Of course, "reflected Brownian motion" is a perfectly good name for Y , and mathematicians understandably felt that (2) was a more natural definition than (1), so Z has come to be known as "an alternative representation of reflected Brownian motion." But the word "reflection" is completely inappropriate as a description of the mapping embodied in (1), and it is this mapping with which one begins in applications (see Chapter 2). Moreover, we are generally interested in the situation where X is a Brownian motion with drift. Then Y and Z do *not* have the same distribution, but most authors, including myself in previous work, have persisted in calling Z "reflected Brownian motion." This terminology has even been extended to higher dimensions, where one encounters mysterious phrases like "Brownian motion with oblique reflection at the boundary." (Problem 13 of Chapter 5 describes a process that is usually characterized in this way.)

Throughout this book, Z is called *regulated Brownian motion*. With this terminology, Levy's theorem says that regulated Brownian motion and reflected Brownian motion have the same distribution in the driftless case. The mapping that carries X into Z is called the *one-sided regulator*, and we say that Z has a *lower control barrier* at zero. This terminology is motivated and extended in Chapter 2. To repeat an earlier point, I propose this new terminology not just because the old name lacks descriptive precision, but because it is confusing and impedes the process of passing theory through to applications.

The term *flow system* is used here to describe the type of physical system studied in queuing and inventory theory. One canonical example is a production system where material flows through one or more manufacturing stages and then to end users upon demand. Most of conventional queuing

and inventory theory is concerned with *buffered stochastic flow*. In the simplest models, there is an input flow and an output flow, such as production and demand, and each may involve stochastic variability. Thus input and output are not perfectly matched over short time intervals, and system performance can be improved by the provision of intermediate buffer storage. In designing and operating such systems, there are economic tradeoffs among the cost of capacity, the costs associated with storage, and the economic benefits associated with system throughput. One can only address those tradeoffs in the context of a stochastic model. An advantage of the term *stochastic flow system* is that it emphasizes the positive purpose for which the system exists, whereas queues, inventories, congestion, and storage are all undesirable aspects of system performance.

A major theme here is the role of regulated Brownian motion as a model of buffered stochastic flow. During the 1960s and early 1970s there was a burst of activity on approximate analysis of queuing systems. Most of this concerned the behavior of such systems under what Kingman (1961) called heavy traffic conditions. (In Chapter 2, the term *balanced loading* is used to describe essentially the same conditions.) The principal conclusion from this work was that certain familiar and relatively intractable queuing processes, if properly normalized, behave in heavy traffic like one-dimensional Brownian motion with a lower control barrier at zero. This was shown quite explicitly by the formal limit theorems of Iglehart-Whitt (1970), but similar words could be used to characterize the conclusions reached in parallel work by Newell (1965) and Gaver (1968) on analytical diffusion approximations. Heavy traffic limit theorems, which serve to justify the use of regulated Brownian motion as a flow system model, will not be discussed here, but readers may consult Whitt (1974) for a survey of that theory. I have tried to explain the sample path behavior of regulated Brownian motion in such a way that its role as a model will be obvious. Most of the effort here is devoted to computing quantities of interest in flow system applications. The results developed in Chapter 5 of this book are largely the same as those presented in Chapter 2 of Newell (1979), but the general approach is quite different.

With respect to mathematical methods, this book emphasizes the Ito stochastic calculus. If one has a probabilistic model built from Brownian motion, such as the process Z defined by (1), then all the interesting associated quantities will be solutions of certain differential equations. For example, in this book I wish to compute expected discounted costs for various processes as functions of the starting state. To calculate such a quantity, what differential equation must be solved, and what are the appropriate auxiliary conditions? Using the celebrated Ito formula, one can answer such questions systematically, which allows one to recast the original problem in

purely analytic terms. Many problems can be solved by direct probabilistic means, such as the martingale methods of Chapter 3, but to solve really hard problems it is necessary to have command of both probabilistic and analytic methods. Thus I believe that Ito's formula is the most important single tool for analysis of Brownian motion and related processes.

The book is organized as follows. In Chapters 1 and 3 the basic properties of Brownian motion are summarized and various standard formulas are derived. Chapter 2, which is very nearly independent of Chapter 1, introduces the basics of flow system modeling. Specific topics discussed there are (a) the regulator maps that underlie the simplest flow system models, (b) discounted measures of system performance, and (c) the role of regulated Brownian motion as a model of buffered stochastic flow. Along the way a simple but illuminating problem of flow system optimization is introduced. This involves first a static capacity decision and then dynamic inventory policy for a manufacturing operation.

Chapter 4 is devoted to the Ito calculus for Brownian motion, emphasizing Ito's formula and its various generalizations. All results are stated in precise mathematical terms, but the major proofs are only sketched. This chapter presents in compact form every aspect of the Ito calculus that I have found valuable in applied probability, and I hope it will be a useful reference for researchers in the field. Chapter 5 then presents a systematic analysis of one-dimensional regulated Brownian motion. Relying heavily on Ito's formula, both discounted performance measures and the steady-state distribution of the process are calculated. Chapter 6 is devoted to a certain highly structured problem in the optimal control of Brownian motion. This problem, motivated by flow system applications, involves a discounted linear cost structure and a nonnegativity constraint on the state variable. The optimal policy is found to involve a lower control barrier at zero and upper control barrier at b , where b is the unique solution of a certain equation. Thus optimization leads to regulated Brownian motion as a system model.

One of my primary objectives in writing this book has been to show exactly why and how Ito's formula is so useful for solving concrete problems. Chapter 4, 5, and 6, together with their problems, have been structured with this goal in mind. I hope that even readers who have no intrinsic interest in stochastic flow systems will find that the applications discussed here enrich their appreciation for the general theory. As with most mathematical subjects, one cannot acquire operational command of the Ito calculus without doing problems, and I believe the problems collected in Chapters 4 and 5 are good ones from this standpoint.

Finally, in Chapter 7 I return to the system optimization problem introduced earlier in Chapter 2. Using results from Chapter 6, the manufacturer's two-stage decision problem is recast as one of optimizing the parameters of a

regulated Brownian motion. Numerical solutions are worked out with various data sets, and several quick-and-dirty approximations are discussed. Although the problem is not one of realistic complexity, I feel that this extended numerical example closes the discussion of flow system modeling with a gratifying tone of concreteness.

Readers are advised to begin with at least a quick look at the appendices. These serve not only to review prerequisite results but also to set notation and terminology. References are collected at the end of each chapter and appendix. I have made no attempt to compile comprehensive bibliographies on any of the subjects discussed nor to suggest the relative contributions of different authors through frequency of citation.

I use §4 when referring to Section 4 of the current chapter, whereas §2.4 refers to Section 4 of Chapter 2. Equations and results are combined in a single numbering system within each section of each chapter. A designation such as (4) refers to the current section, and (2.4) refers to (4) of §2, and (6.2.4) refers to (4) of §6.2. Similarly, §A.2 means §2 of Appendix A and (A.2.4) refers to (4) of §A.2. Chapters 1 to 5 all conclude with a list of Problems and Complements. These are to be viewed as an integral part of the text, rather than optional material. In a similar way, Problem 7 refers to the current chapter, whereas Problem 4.7 is Problem 7 of Chapter 4.

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Notation and Terminology

The expression “ $A \equiv B$ ” means that A is equal to B as a matter of definition. In some sentences, the expression should be read “ A , which is equal by definition to B ,” Conditional expectations are defined only up to an equivalence. Equations involving conditional expectations, or any other random variables, should be interpreted in the almost sure sense. The terms *positive* and *increasing* are used in the weak sense, as opposed to *strictly positive* and *strictly increasing*. The equation

$$P\{X \in dx\} = f(x) dx$$

means that f is a density function for the random variable X . That is,

$$P\{X \in A\} = \int_A f(x) dx$$

for any Borel set A . In the usual way, 1_A denotes the indicator function of a set A , which equals 1 on A and equals zero elsewhere. If $(\Omega, \mathcal{F}, P)$ is a probability space and $A \in \mathcal{F}$, then 1_A is described as an indicator random variable or as the indicator of event A . To specify the time at which a stochastic process X is observed, I may write either X_t or $X(t)$ depending on the situation. On esthetic grounds, I prefer the former notation, but the latter is obviously superior when one must write expressions like $X(T_1 + T_2)$.

Let f be an increasing continuous function on $[0, \infty)$. We say that f increases at a point $t > 0$ if $f(t + \epsilon) > f(t - \epsilon)$ for all $\epsilon > 0$. In this case, t is said to be a *point of increase* for f . Now let g be another continuous function on $[0, \infty)$ and consider the statement

(*) f increases only when $g = 0$.

This means that $g_t = 0$ at every point t where f increases. Many statements of the form (*) appear in this book, and readers will find this terminology to be efficient if somewhat cryptic. The following is a list of symbols that are used with a single meaning throughout the book. Section numbers, when

given, locate either the definition of the symbol or the point of its first appearance (assuming that the appendices are read first).

$\square$	end of proof
$\vee$ and $\wedge$	maximum and minimum
$x^+ \equiv x \vee 0$	positive part of x
$x^- \equiv (-x)^+$	negative part of x
R	the real line
$\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$	filtration (§A.1)
$\mathcal{B}$	Borel σ -algebra on R (§A.2)
$\mathcal{B}[0, \infty)$	Borel σ -algebra on $[0, \infty)$ (§A.2)
$\mathcal{F}_\infty$	§A.2
$C \equiv C[0, \infty)$	§A.2
$\mathcal{C}$	Borel σ -algebra on C (§A.2)
$N(\mu, \sigma^2)$	normal distribution (§1.1)
$V_\beta(t)$	Wald martingale (§1.5)
$\Phi(x)$	$N(0, 1)$ distribution function (§1.6)
P_x and E_x	§3.0
$\Gamma f \equiv \mu f' + \frac{1}{2} \sigma^2 f''$	§3.2
$\alpha_*(\lambda)$ and $\alpha^*(\lambda)$	§3.2
$\psi_*(x)$ and $\psi^*(x)$	§3.2
$\theta_*(x)$ and $\theta^*(x)$	§3.2
$E(X; A)$	partial expectation (§3.2)
H	§4.0
H^2	§4.1
RCLL	right-continuous with left limits (§4.6)

The last section of Appendix B discusses notational conventions for Riemann–Stieltjes integrals. As the reader will see, my general rule is to suppress the arguments of functions appearing in such integrals whenever possible. The same guiding principle is used in Chapters 4 to 6 with respect to stochastic integrals. For example, I write

$$\int_0^t X dW \quad \text{rather than} \quad \int_0^t X(s) dW(s)$$

to denote the stochastic integral of a process X with respect to a Brownian motion W . The former notation is certainly more economical, and it is also

more correct mathematically, but my slavish adherence to the guiding principle may occasionally cause confusion. As an extreme example, consider the expression

$$\int_0^T e^{-\lambda t} (\Gamma - \lambda) f(Z) dg(X + L - U),$$

where λ is a constant, Γ is a differential operator, f and g are functions, and Z , X , L , and U are processes. This signifies the stochastic integral over $[0, T]$ of a process that has value $\exp(-\lambda t)[\Gamma f(Z_t) - \lambda f(Z_t)]$ at time t with respect to a process that has value $g(X_t + L_t - U_t)$ at time t .

Acknowledgments

This book grew out of lecture notes for a course entitled “Stochastic Calculus with Applications,” which I taught five times at Stanford and once while visiting Northwestern University during the 1982–1983 academic year. A substantial portion of the book was written during my stay at Northwestern, and I would like to thank that institution for its kind hospitality. In particular, Erhan Çinlar read virtually all of the first draft and made many valuable suggestions. Among students who took the course, Peter Glynn, Richard Pitbladdo, Tom Sellke, and Ruth Williams all made suggestions whose influence remains visible in the final version. One of the best things about university life is contact with students of their caliber. Ruth Williams and Avi Mandelbaum also offered helpful comments on portions of a later draft and have been invaluable mathematical consultants. Bill Peterson, my research assistant during preparation of the final manuscript, has suggested numerous stylistic improvements and has caught a disconcerting number of errors.

All of Chapter 6 and portions of other chapters are based on papers that I have written with Marty Reiman, Tom Sellke, Michael Taksar, and Hamish Taylor. I would like to acknowledge the contribution that these colleagues have involuntarily made to the current work and to my education over the years. Thanks are also due to the Stanford Graduate School of Business and the Engineering Division of the National Science Foundation for their support of the work from which I have drawn.

There are many other colleagues, former students, and former teachers whose influence can be found on these pages, but several deserve special mention. As a graduate student I learned about regulated Brownian motion and its role in queuing theory from Don Iglehart. Most of the material in Chapters 1 and 3 I originally learned in a course from Dave Siegmund. My original introduction to stochastic calculus (and a lot of other things) came from Dave Kreps, and it was Larry Shepp and Rick Durrett who told me about Tanaka’s Formula and Brownian local time. For bringing these things into my life, I thank them all. Finally, I wish to acknowledge that Figure 2.3, which also graces the jacket of this book, is stolen from Ward Whitt’s Ph.D. thesis. It is a picture worth a thousand words.

J.M.H.

CHAPTER 1

Brownian Motion

The first four sections of this chapter are devoted to the definition of Brownian motion (the mathematical object, not the physical phenomenon) and a compilation of its basic properties. The properties in question are quite deep and readers will be referred elsewhere for proofs. Sections 5 through 9 are devoted to the derivation of further properties and to calculation of several interesting distributions associated with Brownian motion. Before proceeding readers are advised to at least browse through Appendices A and B, which explain several important conventions regarding notation and terminology.

§1. WIENER'S THEOREM

A stochastic process X is said to have *independent increments* if the random variables $X(t_1) - X(t_0), \dots, X(t_n) - X(t_{n-1})$ are independent for any $n \geq 1$ and $0 \leq t_0 < \dots < t_n < \infty$. It is said to have *stationary independent increments* if moreover the distribution of $X(t) - X(s)$ depends only on $t - s$. Finally, we write $Z \sim N(\mu, \sigma^2)$ to mean that the random variable Z has the normal distribution with mean μ and variance σ^2 . A *standard Brownian motion*, or *Wiener process*, is then defined as a stochastic process X having continuous sample paths, stationary independent increments, and $X(t) \sim N(0, t)$. Thus, in our terminology, a standard Brownian motion starts at level zero almost surely. A process Y will be called a (μ, σ) Brownian motion if it has the form $Y(t) = Y(0) + \mu t + \sigma X(t)$, where X is a Wiener process and $Y(0)$ is independent of X . It follows that $Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s)$. We call μ and σ^2 the *drift* and *variance* of Y , respectively. The term *Brownian motion*, without modifier, will be used to embrace all such processes Y .

There remains the question of whether standard Brownian motion exists and whether it is in any sense unique. That is the subject of Wiener's theorem. For its statement, let $\mathcal{C}$ be the Borel σ -algebra on $C = C[0, \infty)$ as in §A.2, and let X be the coordinate process on C as in §A.3. The following is proved in the setting of $C[0, 1]$ in §9 of Billingsley (1968); the extension to $C[0, \infty)$ is essentially trivial.

(1) Wiener's Theorem. There exists a unique probability measure P on $(C, \mathcal{C})$ such that the coordinate process X on $(C, \mathcal{C}, P)$ is a standard Brownian motion.

This P will be referred to hereafter as the *Wiener measure*. It is left as an exercise to show that a continuous process is a standard Brownian motion if and only if its distribution (see §A.2) is the Wiener measure. When combined with (1), this shows that standard Brownian motion exists and is unique *in distribution*. No stronger form of uniqueness can be hoped for because the definitive properties of standard Brownian motion refer only to the distribution of the process. Before concluding this section, we record one more important result. See Chapter 12 of Breiman (1968) for a proof of this theorem.

(2) Theorem. If Y is a continuous process with stationary independent increments, then Y is a Brownian motion.

This beautiful theorem shows that Brownian motion can actually be defined by stationary independent increments and path continuity alone, with normality following as a consequence of these assumptions. This may do more than any other characterization to explain the significance of Brownian motion for probabilistic modeling.

§2. A RICHER SETTING

With an eye toward future requirements, we now introduce the idea of a Brownian motion *with respect to a given filtration*. Let $(\Omega, \mathbb{F}, P)$ be a filtered probability space in the sense of §A.1, and let X be a continuous process on this space. We say that X is a (μ, σ) Brownian motion with respect to $\mathbb{F}$, or simply a (μ, σ) Brownian motion on $(\Omega, \mathbb{F}, P)$, if

- (1) X is adapted,
- (2) $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t$, and
- (3) X is a (μ, σ) Brownian motion in the sense of §1.

Roughly speaking, (1) and (2) say that $\mathcal{F}_t$ contains complete information about the history of X up to time t , but no information at all about the evolution of X after t . For a specific example, one may take the canonical space of §A.3 with P the Wiener measure. In that case, X is a *standard* Brownian motion on $(\Omega, \mathbb{F}, P)$. Throughout the remainder of this chapter, we take X to be a (μ, σ) Brownian motion *with starting state zero* on some filtered probability space $(\Omega, \mathbb{F}, P)$. Note that the requirement $X_0 = 0$ is incremental to (1) to (3).

§3. QUADRATIC VARIATION AND LOCAL TIME

One of the best known properties of Brownian motion is that almost all its sample paths have infinite variation over any time interval of positive length. Thus Brownian sample paths are emphatically *not* VF functions (see §B.2). In contrast to this negative result, a very sharp positive statement can be made about the so-called quadratic variation of Brownian paths. Let $f: [0, \infty) \rightarrow R$ be arbitrary and fix $t > 0$. If the limit

$$(1) \quad q_t \equiv \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \left[f\left(\frac{(k+1)t}{2^n}\right) - f\left(\frac{kt}{2^n}\right) \right]^2$$

exists (including $+\infty$ as a possibility), then we call q_t the quadratic variation of f over $[0, t]$. It should be emphasized that this limit, unlike the one defining ordinary variation, need not exist, but the following is one case where it obviously does. (The proof of this statement is left as an exercise.)

(2) Proposition. If f is a continuous VF function, then $q_t = 0$ for all $t \geq 0$.

To discuss the quadratic variation of Brownian paths, let us define $Q_t(\omega)$ by (1) with $X(\omega)$ in place of f , assuming for the moment that the limit exists. The following proposition is proved in most standard texts, but a particularly thorough treatment of this and related properties is given by Freedman (1971).

(3) Proposition. For almost every $\omega \in \Omega$ we have $Q_t(\omega) = \sigma^2 t$ for all $t \geq 0$.

Three increasingly surprising implications of (3) are as follows. First, the quadratic variation Q_t exists for almost all Brownian paths and all $t \geq 0$.

Second, it is not zero if $t > 0$, and hence X almost surely has infinite ordinary variation over $[0, t]$ by (2). Finally, the quadratic variation of X does not depend on ω ! Readers should note that the dyadic partitioning scheme by which we compute quadratic variation is independent of ω ; Freedman (1971) discusses the importance of this restriction.

It would be difficult to overstate the significance of Proposition (3). We shall see later that it contains the essence of Ito's formula, and that Ito's formula is the key tool for analysis of Brownian motion and related processes. Although a complete proof of (3) would carry us too far afield, there are some easy calculations that at least help to make this critical result plausible. If f is replaced by X in (1), then the *expected value* of the sum on the right side is

$$(4) \quad \begin{aligned} \sum_{k=0}^{2^n-1} E \left\{ \left[X \left(\frac{(k+1)t}{2^n} \right) - X \left(\frac{kt}{2^n} \right) \right]^2 \right\} \\ = 2^n \left\{ \left[\mu \left(\frac{t}{2^n} \right) \right]^2 + \sigma^2 \left(\frac{t}{2^n} \right) \right\} \\ = \mu^2 t^2 2^{-n} + \sigma^2 t \rightarrow \sigma^2 t \text{ as } n \rightarrow \infty. \end{aligned}$$

Similarly, using the independent increments of X , one may calculate explicitly the variance of the sum. (This calculation is left as an exercise.) The variance is found to vanish as $n \rightarrow \infty$, proving that the sums converge to $\sigma^2 t$ in the L^2 sense as $n \rightarrow \infty$. Proposition (3) says that they also converge almost surely.

Another nice feature of Brownian paths arises in conjunction with the *occupancy measure* of the process. For each $\omega \in \Omega$ and $A \in \mathcal{B}$ (the Borel σ -algebra on R) let

$$\nu(t, A, \omega) \equiv \int_0^t 1_A(X_s(\omega)) ds, \quad t \geq 0,$$

with the integral defined in the Lebesgue sense. Thus $\nu(t, A, \cdot)$ is a random variable representing the amount of time spent by X in the set A up to time t , and $\nu(t, \cdot, \omega)$ is a positive measure on $(R, \mathcal{B})$ having total mass t ; this is the occupancy measure alluded to above. The following theorem, one of the deepest of all results relating to Brownian motion, says that the occupancy measure is absolutely continuous with respect to Lebesgue measure and has a smooth density. See §7.2 of Chung-Williams (1983) for a proof.

(5) Theorem. There exists $l: [0, \infty) \times R \times \Omega \rightarrow R$ such that, for almost every ω , $l(t, x, \omega)$ is jointly continuous in t and x and

$$\nu(t, A, \omega) = \int_A l(t, x, \omega) dx \quad \text{for all } t \geq 0 \text{ and } A \in \mathcal{B}.$$

The most difficult and surprising part of this result is the continuity of l in x , a smoothness property that testifies to the erratic behavior of Brownian paths. (Consider the occupancy measure corresponding to a continuously differentiable sample path. You will see that it does not have a continuous density at points x that are achieved as local maxima or minima of the path.) From (5) it follows that, for almost all ω ,

$$(6) \quad l(t, x, \omega) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon, x+\varepsilon]}(X_s(\omega)) ds,$$

for all $t \geq 0$ and $x \in R$. Consequently $l(\cdot, x, \omega)$ is a continuous increasing function that increases only at time points t where $X(t, \omega) = x$. The stochastic process $l(\cdot, x, \cdot)$ is called the *local time* of X at level x .

(7) Proposition. If $u: R \rightarrow R$ is bounded and measurable, then for almost all ω we have

$$(8) \quad \int_0^t u(X_s(\omega)) ds = \int_R u(x) l(t, x, \omega) dx, \quad t \geq 0.$$

Proof. If u is the indicator 1_A for some $A \in \mathcal{B}$, then (8) follows from (5). Thus (8) holds for all simple functions u (finite linear combinations of indicators). For any positive, bounded, measurable u we can construct simple functions $\{u_n\}$ such that $u_n(x) \uparrow u(x)$ for almost every x (Lebesgue measure). Because (8) is valid for each u_n , it is also valid for u by the monotone convergence theorem. Moreover, the right side of (8) is finite because $l(t, \cdot, \omega)$ has compact support. The proof is concluded by the observation that every bounded, measurable function is the difference of two positive, bounded, measurable functions. $\square$

§4. STRONG MARKOV PROPERTY

Remember that X is a (μ, σ) Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, P)$. When we speak of stopping times (see §A.1), implicit reference is being made to the filtration $\mathcal{F}$. Here and later we write $T < \infty$ as shorthand for the more precise statement $P\{T < \infty\} = 1$.

(1) Theorem. Let $T < \infty$ be a stopping time, and define $X_t^* = X_{T+t} - X_T$ for $t \geq 0$. Then X^* is a (μ, σ) Brownian with starting state zero and X^* is independent of $\mathcal{F}_T$.

Let $\mathcal{F}^*$ be the smallest σ -algebra with respect to which all the random variables $\{X_t^*, t \geq 0\}$ are measurable. The last phrase of the theorem means that $\mathcal{F}_T$ and $\mathcal{F}^*$ are independent σ -algebras. Theorem (1) is proved in a slightly more restrictive setting by Breiman (1968), but his proof extends to our situation without trouble. This result articulates the strong Markov property in a form unique to Brownian motion. See Chapter 3 for an equivalent statement that suggests more clearly what is meant by a strong Markov process in general.

§5. BROWNIAN MARTINGALES

Recall that $X_t - X_s$ is independent of $\mathcal{F}_s$ for $s \leq t$ by (2.2). If $\mu = 0$, then we have

$$(1) \quad E(X_t - X_s | \mathcal{F}_s) = E(X_t - X_s) = 0$$

and

$$(2) \quad E[(X_t - X_s)^2 | \mathcal{F}_s] = E[(X_t - X_s)^2] = \sigma^2(t - s).$$

Obviously (1) can be restated as

$$(3) \quad E(X_t | \mathcal{F}_s) = X_s,$$

and then the left side of (2) reduces to

$$\begin{aligned} (4) \quad E[(X_t - X_s)^2 | \mathcal{F}_s] &= E(X_t^2 | \mathcal{F}_s) - 2E(X_t X_s | \mathcal{F}_s) + X_s^2 \\ &= E(X_t^2 | \mathcal{F}_s) - 2X_s E(X_t | \mathcal{F}_s) + X_s^2 \\ &= E(X_t^2 | \mathcal{F}_s) - X_s^2. \end{aligned}$$

Substituting (4) into (2) and rearranging terms gives

$$(5) \quad E(X_t^2 - \sigma^2 t | \mathcal{F}_s) = X_s^2 - \sigma^2 s.$$

Now (3) and (5) may be restated as follows.

(6) Proposition. If $\mu = 0$, then X and $\{X_t^2 - \sigma^2 t, t \geq 0\}$ are martingales on $(\Omega, \mathbb{F}, P)$.

From (2.1) to (2.3) we know that the conditional distribution of $X_t - X_s$ given $\mathcal{F}_s$ is $N(\mu(t - s), \sigma^2(t - s))$. From this it follows that

$$(7) \quad E[\exp\{\beta(X_t - X_s)\} | \mathcal{F}_s] = \exp\{\mu\beta(t - s) + \frac{1}{2}\sigma^2\beta^2(t - s)\}$$

for any $\beta \in R$ and $s < t$. Now let

$$(8) \quad q(\beta) \equiv \mu\beta + \frac{1}{2}\sigma^2\beta^2, \quad \beta \in R$$

(the letter q is mnemonic for *quadratic*) and note that (7) can be rewritten as

$$(9) \quad E[\exp\{\beta(X_t - X_s) - q(\beta)(t - s)\} | \mathcal{F}_s] = 1.$$

From (9) it is immediate that $E[V_\beta(t) | \mathcal{F}_s] = V_\beta(s)$, where

$$(10) \quad V_\beta(t) \equiv \exp\{\beta X_t - q(\beta)t\}, \quad t \geq 0.$$

Thus we arrive at the following.

(11) Proposition. V_β is a martingale on $(\Omega, \mathbb{F}, P)$ for each $\beta \in R$.

Hereafter we shall refer to V_β as the *Wald martingale* with dummy variable β . Readers will find that it plays a central role in the calculations of Chapter 3.

§6. A JOINT DISTRIBUTION (REFLECTION PRINCIPLE)

Let $M_t \equiv \sup\{X_s, 0 \leq s \leq t\}$ and then define the joint distribution function

$$(1) \quad F_t(x, y) \equiv P\{X_t \leq x, M_t \leq y\}.$$

Because $X_0 = 0$ by hypothesis, one need only calculate $F_t(x, y)$ for $y \geq 0$ and $x \leq y$; the discussion is hereafter restricted to (x, y) pairs satisfying these two conditions. We shall compute F for *standard* Brownian motion in this section and then extend the calculation to general μ and σ in §8. Fixing $\mu = 0$ and $\sigma = 1$ throughout this section, note first that

$$\begin{aligned} (2) \quad F_t(x, y) &= P\{X_t \leq x\} - P\{X_t \leq x, M_t > y\} \\ &= \Phi(xt^{-1/2}) - P\{X_t \leq x, M_t > y\}, \end{aligned}$$

where $\Phi(\cdot)$ is the $N(0,1)$ distribution function. Now the term $P\{X_t \leq x, M_t > y\}$ can be calculated heuristically using the so-called reflection principle (note that the restriction $\mu = 0$ is critical here) as follows: For every sample path of X that hits level y before time t but finishes below level x at time t , there is another equally probable path (shown by the dotted line in Figure 1) that hits y before t and then travels *upward* at least $y - x$ units to finish above level $y + (y - x) = 2y - x$ at time t . Thus

$$(3) \quad P\{X_t \leq x, M_t > y\} = P\{X_t \geq 2y - x\} \\ = P\{X_t \leq x - 2y\} = \Phi((x - 2y)t^{-1/2}).$$

This argument is not rigorous, of course, but it can be made so using the strong Markov property of §4. Let T be the first t at which $X_t = y$, and define X^* as in Theorem (4.1). From (4.1) it follows that

$$P\{X_t \leq x, M_t > y\} = P\{T < t, X^*(t - T) \leq x - y\} \\ = P\{T < t, X^*(t - T) \geq y - x\}.$$

(The strong Markov property is needed to justify the *second* of these equalities.) By definition $X^*(t - T) = X(t) - y$ and thus we arrive at (3). Combining (2) and (3) gives the following proposition. For the corollary, differentiate with respect to x .

(4) Proposition. If $\mu = 0$ and $\sigma = 1$, then

$$(5) \quad P\{X_t \leq x, M_t \leq y\} = \Phi(xt^{-1/2}) - \Phi((x - 2y)t^{-1/2}).$$

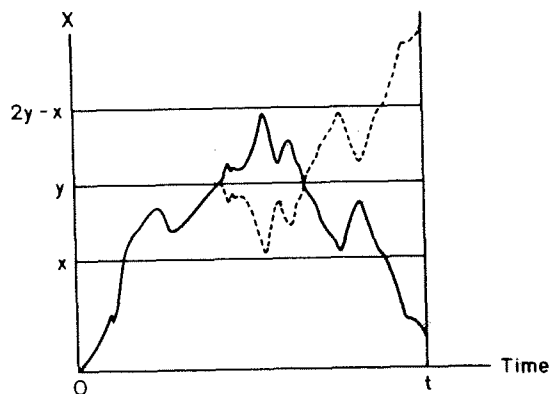


Figure 1. The reflection principle.

(6) Corollary. $P\{X_t \in dx, M_t \leq y\} = g_t(x, y) dx$, where

$$(7) \quad g_t(x, y) \equiv [\phi(xt^{-1/2}) - \phi((x - 2y)t^{-1/2})]t^{-1/2}$$

and $\phi(z) \equiv (2\pi)^{-1/2} \exp(-z^2/2)$ is the $N(0,1)$ density function.

§7. CHANGE OF DRIFT AS CHANGE OF MEASURE

For this section, let $T > 0$ be fixed and deterministic, and restrict X to the time domain $[0, T]$. Starting with the (μ, σ) Brownian motion $X = \{X_t, 0 \leq t \leq T\}$ on $(\Omega, \mathbb{F}, P)$, suppose we want to construct a $(\mu + \theta, \sigma)$ Brownian motion, also with time domain $[0, T]$. One approach is to keep the original space $(\Omega, \mathbb{F}, P)$ and define a new process $Z_t(\omega) = X_t(\omega) + \theta t$, $0 \leq t \leq T$. Then Z is a $(\mu + \theta, \sigma)$ Brownian motion on $(\Omega, \mathbb{F}, P)$.

Another approach is to keep the original process X and change the probability measure. The idea is to replace P by some other probability measure P^* such that X is a $(\mu + \theta, \sigma)$ Brownian motion on $(\Omega, \mathbb{F}, P^*)$. In this section, we shall do just that. A positive random variable ξ will be displayed, and then P^* will be defined via

$$(1) \quad P^*(A) = \int_A \xi(\omega) P(d\omega), \quad A \in \mathcal{F}.$$

It is usual to express (1) in the more abstract form $dP^* = \xi dP$ and to call ξ the *density* (or Radon-Nikodym derivative) of P^* with respect to P . It will be seen that $P\{\xi > 0\} = 1$, so P and P^* are *equivalent measures*, meaning that $P^*(A) = 0$ if and only if $P(A) = 0$ (the two measures have the same *null sets*).

For the first two propositions below, P^* can be *any* probability measure related to P via (1) with $P\{\xi > 0\} = 1$. It is, of course, necessary that $E(\xi) = \int \xi dP = 1$, for otherwise P^* would not be a *probability measure*. It will be useful to denote by E^* the expectation operator associated with P^* , meaning that

$$(2) \quad E^*(f) \equiv \int_{\Omega} f dP^* \equiv \int_{\Omega} (f \cdot \xi) dP \equiv E(\xi f)$$

for measurable functions $f: \Omega \rightarrow \mathbb{R}$ such that $E|\xi f| < \infty$. Also, let $\xi(t) \equiv E(\xi | \mathcal{F}_t)$ for $0 \leq t \leq T$, so that $\{\xi(t), 0 \leq t \leq T\}$ is a (strictly positive) martingale on $(\Omega, \mathbb{F}, P)$. The proofs of the following propositions require little more than the definitions of conditional expectation and martingale, respectively; they are left as exercises.

(3) Proposition. Let f be a random variable with $E^*|f| < \infty$. Then $E^*(f|\mathcal{F}_t) = E(\xi f|\mathcal{F}_t)/\xi(t)$, $0 \leq t \leq T$.

(4) Corollary. Let $Z = \{Z(t), 0 \leq t \leq T\}$ be a process adapted to $\mathbb{F}$. Then Z is a martingale on $(\Omega, \mathbb{F}, P^*)$ if and only if $\{Z(t)\xi(t), 0 \leq t \leq T\}$ is a martingale on $(\Omega, \mathbb{F}, P)$.

Recall now the definitions of $q(\beta)$ and $V_\beta(t)$ from §5. Given $\theta \in R$, the particular density ξ that will meet our requirements is

$$(5) \quad \xi = V_\gamma(T), \text{ where } \gamma \equiv \theta/\sigma^2.$$

Before the main theorem is proved, a few observations are appropriate. First, $P\{\xi > 0\} = 1$ as claimed earlier. Second, V_γ is a martingale on $(\Omega, \mathbb{F}, P)$ by (5.11), so

$$(6) \quad E(\xi|\mathcal{F}_0) = E[V_\gamma(T)|\mathcal{F}_0] = V_\gamma(0) = 1,$$

implying $E(\xi) = 1$ as required. (The last equality in (6) follows from the fact that $X_0 = 0$ by assumption.) More generally,

$$(7) \quad E(\xi|\mathcal{F}_t) = E[V_\gamma(T)|\mathcal{F}_t] = V_\gamma(t), \quad 0 \leq t \leq T.$$

Comparing (7) with the earlier definition of $\xi(t)$, we see that, when ξ is defined by (5),

$$(8) \quad \xi(t) = V_\gamma(t), \quad 0 \leq t \leq T.$$

(9) Change of Measure Theorem. Given $\theta \in R$, let ξ and P^* be defined by (5) and (1), respectively. Then X is a $(\mu + \theta, \sigma)$ Brownian motion on $(\Omega, \mathbb{F}, P^*)$.

Proof. Let $\beta \in R$ be arbitrary and define

$$(10) \quad V_\beta^*(t) \equiv \exp[\beta X_t - q^*(\beta)t], \quad 0 \leq t \leq T,$$

where

$$(11) \quad q^*(\beta) \equiv (\mu + \theta)\beta + \frac{1}{2}\sigma^2\beta^2.$$

By Proposition (5.11), a *necessary* condition for the desired conclusion is that

$$(12) \quad V_\beta^* \text{ is a martingale on } (\Omega, \mathbb{F}, P^*).$$

In Problem 5 a converse of (5.11) will be proved, this implying that (12) is also *sufficient* for the desired conclusion. (More precisely, the proof will be sketched and readers will be asked to provide details.) Now (4) shows that (12) is equivalent to the requirement that

$$(13) \quad \{V_\beta^*(t)\xi(t), 0 \leq t \leq T\} \text{ is a martingale on } (\Omega, \mathbb{F}, P).$$

From (10) and (8) we have that

$$(14) \quad \begin{aligned} V_\beta^*(t)\xi(t) &= \exp[\beta X_t - q^*(\beta)t] \exp(\gamma X_t - q(\gamma)t) \\ &= \exp[(\beta + \gamma)X_t - \psi(\beta)t], \end{aligned}$$

where

$$(15) \quad \psi(\beta) \equiv q^*(\beta) + q(\gamma) = [(\mu + \theta)\beta + \frac{1}{2}\sigma^2\beta^2] + [\mu\gamma + \frac{1}{2}\sigma^2\gamma^2].$$

Using the fact that $\gamma \equiv \theta/\sigma^2$, readers can now verify that $\psi(\beta) = \mu(\beta + \gamma) + \sigma^2(\beta + \gamma)^2/2 \equiv q(\beta + \gamma)$. Thus (14) says that $V_\beta^*(t)\xi(t) = V_{\beta+\gamma}(t)$. Finally, $V_{\beta+\gamma}$ is a martingale on $(\Omega, \mathbb{F}, P)$ by (5.11), so (13) holds and the proof is complete. $\square$

It should be emphasized that the change of measure theorem is only valid when one views X as a process with finite time horizon. However, it can be generalized to the case where T is a stopping time; also many other generalizations are known. See Chapter 6 of Liptser–Shiryayev (1977) for a taste of the more general theory.

§8. A HITTING TIME DISTRIBUTION

Returning to the analysis begun in §6, we now use the change of measure theorem to calculate the joint distribution of X_t and M_t in generality.

(1) Proposition. For general values of μ and σ we have

$$(2) \quad P\{X_t \in dx, M_t \leq y\} = f_t(x, y) dx,$$

where

$$(3) \quad f_t(x, y) = (1/\sigma) \exp(\mu x/\sigma^2 - \mu^2 t/2\sigma^2) g_t(x/\sigma, y/\sigma)$$

and $g_t(\cdot, \cdot)$ is defined by (6.7).

Proof. Only the case $\sigma = 1$ will be treated here; the extension to general σ is accomplished by a straightforward rescaling. Suppose initially that X is a *standard* Brownian motion on $(\Omega, \mathbb{F}, P)$ so that

$$(4) \quad P\{X_t \in dx, M_t \leq y\} = g_t(x, y) dx$$

by (6.6). Now fix $t > 0$, let $\mu \in R$ be arbitrary, set

$$(5) \quad \xi \equiv \exp(\mu X_t - \frac{1}{2} \mu^2 t),$$

and define a new probability measure P^* by taking $dP^* = \xi dP$. The change of measure theorem (7.9) says that $\{X_s, 0 \leq s \leq t\}$ is a $(\mu, 1)$ Brownian motion under P^* , so the desired result (specialized to $\sigma = 1$) is equivalently stated as

$$(6) \quad P^*\{X_t \in dx, M_t \leq y\} = e^{\mu x - \mu^2 t/2} g_t(x, y) dx.$$

To simplify typography in the proof of (6), let us denote by $1(A)$ the random variable that has value 1 on A and value zero otherwise. Using (4),

$$\begin{aligned} P^*\{X_t \leq x, M_t \leq y\} &= E^*[1(X_t \leq x, M_t \leq y)] \\ &= E[\xi \cdot 1(X_t \leq x, M_t \leq y)] \\ &= E[e^{\mu X_t - \mu^2 t/2} 1(X_t \leq x, M_t \leq y)] \\ &= \int_{-\infty}^x e^{\mu z - \mu^2 t/2} P\{X_t \in dz, M_t \leq y\} \\ &= \int_{-\infty}^x e^{\mu z - \mu^2 t/2} g_t(z, y) dz. \end{aligned}$$

Differentiating with respect to x gives (6) as required. $\square$

(7) Corollary. Let $F_t(x, y) \equiv P\{X_t \leq x, M_t \leq y\}$ as in §6. For general values of μ and σ we have

$$(8) \quad F_t(x, y) = \Phi\left(\frac{x - \mu t}{\sigma t^{1/2}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{x - 2y - \mu t}{\sigma t^{1/2}}\right).$$

Proof. Again we treat only the case $\sigma = 1$. By specializing the general formula (3) for f accordingly, we obtain

$$\begin{aligned} (9) \quad F_t(x, y) &= \int_{-\infty}^x f_t(z, y) dz \\ &= e^{-\mu^2 t/2} \int_{-\infty}^x e^{\mu z} t^{-1/2} [\phi(t^{-1/2} z) - \phi(t^{-1/2} (z - 2y))] dz \\ &= e^{-\mu^2 t/2} \int_{-\infty}^0 e^{\mu(x+z)} t^{-1/2} [\phi(t^{-1/2} (z + x)) - \phi(t^{-1/2} (z + x - 2y))] dz \\ &= e^{\mu x - \mu^2 t/2} [\Psi(x) - \Psi(x - 2y)], \end{aligned}$$

where

$$(10) \quad \Psi(x) \equiv \int_{-\infty}^0 t^{-1/2} e^{\mu z} \phi(t^{-1/2} (z + x)) dz.$$

Now let $h(x, t) \equiv t^{-1/2}(x - \mu t)$. Writing out $\phi(\cdot)$ and completing the square in the exponent, we have

$$\begin{aligned} \Psi(x) &= \int_{-\infty}^0 (2\pi t)^{-1/2} \exp\left\{\mu z - \frac{(z^2 + 2xz + x^2)}{2t}\right\} dz \\ &= \int_{-\infty}^0 (2\pi t)^{-1/2} \exp\left\{-\left[\frac{z^2 + 2(x - \mu t)z + (x - \mu t)^2}{2t}\right] + \left(\frac{\mu^2 t}{2} - \mu x\right)\right\} dz \\ &= e^{-\mu x + \mu^2 t/2} \int_{-\infty}^0 (2\pi t)^{-1/2} \exp\{-\frac{1}{2}[t^{-1/2}(z + x - \mu t)]^2\} dz \\ &= e^{-\mu x + \mu^2 t/2} \int_{-\infty}^0 t^{-1/2} \phi(t^{-1/2}(z + x - \mu t)) dz \\ &= e^{-\mu x + \mu^2 t/2} \int_{-\infty}^{h(x, t)} \phi(u) du \\ &= e^{-\mu x + \mu^2 t/2} \Phi(h(x, t)). \end{aligned}$$

Substituting this into (9) gives the desired formula. $\square$

If we define $T(y)$ as the first t at which $X_t = y$ (possibly $+\infty$ if $\mu < 0$), then obviously $T(y) > t$ if and only if $M_t < y$. Letting $x \uparrow y$ in (8) gives

$$(11) \quad \begin{aligned} P\{T(y) > t\} &= P\{M_t < y\} = F_t(y, y) \\ &= \Phi\left(\frac{y - \mu t}{\sigma t^{1/2}}\right) - e^{2\mu y/\sigma^2} \Phi\left(\frac{-y - \mu t}{\sigma t^{1/2}}\right) \end{aligned}$$

for $y > 0$. With this we have calculated explicitly the *one-sided first passage time distribution for Brownian motion with drift*.

§9. REGULATED BROWNIAN MOTION

Remember that $X_0 = 0$ by assumption throughout this chapter. Let us now define an increasing process L and a positive process Z by setting

$$(1) \quad L_t = - \inf_{0 \leq s \leq t} X_s, \quad t \geq 0,$$

$$(2) \quad Z_t \equiv X_t + L_t = \sup_{0 \leq s \leq t} (X_t - X_s), \quad t \geq 0.$$

Later Z will be called *regulated Brownian motion* with a lower control barrier at zero. The very simple representation (2) is specific to the case $X_0 = 0$, but in Chapter 2 a general representation for arbitrary starting state will be developed, and we shall also consider the case of two control barriers. The probabilistic and the analytic theory of regulated Brownian motion will be developed in later chapters.

A slight modification of the arguments used in §6 and §8 gives the joint distribution of X_t and L_t , from which one can obviously calculate the distribution of Z_t . But here is an easier way. Fix $t > 0$ and for $0 \leq s \leq t$ let $X_s^* = X_t - X_{t-s}$. Note that $X^* \equiv \{X_s^*, 0 \leq s \leq t\}$ has stationary, independent increments with $X_0^* = 0$ and $X_s^* \sim N(\mu s, \sigma^2 s)$. Thus X^* is another (μ, σ) Brownian motion with starting state zero. Combining this with (2), we get

$$(3) \quad \begin{aligned} Z_t &= \sup_{0 \leq s \leq t} (X_t - X_s) = \sup_{0 \leq s \leq t} (X_t - X_{t-s}) \\ &= \sup_{0 \leq s \leq t} X_s^* \sim \sup_{0 \leq s \leq t} X_s \equiv M_t. \end{aligned}$$

(Here the symbol $\sim$ denotes equality in distribution.) Thus the distributions of Z_t and M_t coincide for each fixed t , although the distributions of the complete processes Z and M are very different. (For example, M has increasing sample paths, but Z does not.)

The marginal distribution of M was displayed earlier in (8.11). Combining this with (3) gives

$$(4) \quad P\{Z_t \leq z\} = \Phi\left(\frac{z - \mu t}{\sigma t^{1/2}}\right) - e^{2\mu z/\sigma^2} \Phi\left(\frac{-z - \mu t}{\sigma t^{1/2}}\right)$$

for all $t \geq 0$. Thus as $t \rightarrow \infty$,

$$(5) \quad P\{Z_t \leq z\} \rightarrow \begin{cases} 1 - e^{2\mu z/\sigma^2} & \text{if } \mu < 0 \\ 0 & \text{if } \mu \geq 0. \end{cases}$$

For $\mu < 0$, the limit distribution (5) is exponential with mean $\sigma^2/2|\mu|$.

We shall continue the analysis of Z later using the machinery of stochastic calculus. To prepare the way, it will be useful to record some properties of the process L . (Everything said here would apply equally well to M .) It is obviously continuous, but the following well-known proposition shows that L increases in a very jerky fashion.

(6) Proposition. For almost every ω and any $t > 0$, $L(\omega)$ has uncountably many points of increase in $[0, t]$, but the set of all such points has (Lebesgue) measure zero.

Because the sample paths of L increase only on a set of measure zero, they cannot be absolutely continuous; L cannot be expressed as the integral of another process (see §B.1). One cannot speak of the *rate* at which L increases, although its sample paths are continuous and increasing and are therefore VF functions (see §B.2). Because L plays such an important role in this book, the distinction between VF processes and absolutely continuous processes is an important one for us.

PROBLEMS AND COMPLEMENTS

1. Let X be a continuous process with distribution Q (see §A.2). It was stated in §1 that the definitive properties of standard Brownian motion (SBM) involve only the distribution of the process. This means that X is an SBM if and only if Q satisfies certain conditions. Write out in precise mathematical form what those conditions are, and then show that X is an SBM if and only if Q is the Wiener measure. Although this problem requires nothing more than shuffling definitions, it is difficult for those who have never dealt with stochastic processes in abstract terms. It

requires that one understand the general distinction between a stochastic process and its distribution, and the specific distinction between standard Brownian motion and the Wiener measure.

2. Prove Proposition (3.2), which says that a continuous VF function has zero quadratic variation.
3. Calculate the variance of the sum on the left side of (3.4) and show that this vanishes as $n \rightarrow \infty$.
4. Let X be the coordinate process on C as in §A.3 and let $\nu(t, A, \omega)$ be the occupancy measure for X , defined by (3.4). Consider the particular point $\omega \in C$ defined by $\omega(t) = (1 - t)^2$, $t \geq 0$. Fix a time $t > 1$ and describe $\nu(t, \cdot, \omega)$ in precise mathematical terms. Observe that this measure on $(R, \mathcal{B})$ is absolutely continuous (with respect to Lebesgue measure) but its density is not continuous. This substantiates a claim made in §3.
5. Prove (7.3) and (7.4). This is just a matter of verification, using the definitions of conditional expectation and martingale.
6. Let X be a continuous adapted process on some filtered probability space $(\Omega, \mathcal{F}, P)$. Define $V_\beta(t)$ in terms of X via (5.8) and (5.10). The converse of (5.11) that was invoked in proving the change of measure theorem (7.9) is the following: If V_β is a martingale for each $\beta \in R$, then X is a (μ, σ) Brownian motion on $(\Omega, \mathcal{F}, P)$. The problem is to prove this, specializing to the case $\mu = 0$ and $\sigma = 1$. As a first step, observe that X is a $(0, 1)$ Brownian motion on $(\Omega, \mathcal{F}, P)$ if and only if

$$(*) \quad P\{X_{t+s} - X_t \leq x | \mathcal{F}_t\} = \Phi(s^{-1/2} x)$$

for $x \in R$ and $s, t \geq 0$. Then show that $(*)$ is equivalent to

$$E\{\exp \beta(X_{t+s} - X_t) | \mathcal{F}_t\} = e^{\beta^2 s/2}.$$

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CHAPTER 2

Stochastic Models of Buffered Flow

Consider a firm that produces a single durable commodity on a make-to-stock basis. Production flows into a finished goods inventory, and demand that cannot be met from stock on hand is simply lost, with no adverse effect on future demand. The price of the output good is fixed, and demand is viewed as an exogenous source of uncertainty. Similarly, we consider plant, equipment, and work force size to be fixed for now, but there may be uncertainty about actual production quantities because of mechanical failures, worker absenteeism, and so forth. This firm and its market, portrayed schematically in Figure 1, constitute what we call a two-stage flow system. It consists of an input process (production), an output process (demand), and an intermediate buffer storage (the finished goods inventory) that serves to decouple input and output. Many mathematical models of such flow systems have been developed, with some aimed at particular areas of application and some quite abstract in character. For a sampling of these models see Arrow-Scarf-Karlin (1958), Moran (1959), Cox-Smith (1961), and Kleinrock (1976).

The abstract language of input processes, output processes, and storage buffers will be used hereafter, but the content of the buffer will be called inventory, and readers will find that all our examples involve production systems. In this chapter we develop a crude model of buffered flow, making no attempt to portray physical structure beyond that apparent in Figure 1. Actually, two models will be advanced, one with infinite buffer capacity and one with finite capacity. In each case, system flows are represented by continuous stochastic processes. Thus our models have little relevance to systems where individual inventory items are physically or economically

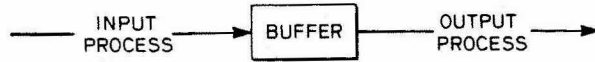


Figure 1. A two-stage flow system.

significant, but for discrete item systems with high-volume flow, the continuity assumption may be viewed as a convenient and harmless idealization.

§1. A SIMPLE FLOW SYSTEM MODEL

Assume that the buffer in Figure 1 has infinite capacity. To model the system, we take as primitive a constant $X_0 \geq 0$ and two increasing, continuous stochastic processes $A = \{A_t, t \geq 0\}$ and $B = \{B_t, t \geq 0\}$ with $A_0 = B_0 = 0$. Interpret X_0 as the initial inventory level, A_t as the cumulative input up to time t , and B_t as the cumulative *potential* output up to time t . In other words, B_t is the total output that can be realized over the time interval $[0, t]$ if the buffer is never empty; more generally, $B_t - B_s$ is the maximum possible output over the interval $(s, t]$. If emptiness does occur, then some of this potential output will be lost. We denote by L_t the amount of potential output lost up to time t because of such emptiness, so *actual* output over $[0, t]$ is $B_t - L_t$. Setting

$$(1) \quad X_t \equiv X_0 + A_t - B_t,$$

the inventory at time t is then given by

$$(2) \quad Z_t \equiv X_0 + A_t - (B_t - L_t) = X_t + L_t.$$

Most of our attention will focus on this *inventory process* $Z = \{Z_t, t \geq 0\}$. It remains to define the lost potential output process L in terms of primitive model elements, and for that we simply assume (or require) that

- (3) L is increasing and continuous with $L_0 = 0$ and
- (4) L increases only when $Z = 0$.

Conditions (3) and (4) together say that output is (by assumption) sacrificed in the minimum amounts consistent with the physical restriction

$$(5) \quad Z_t \geq 0 \quad \text{for all } t \geq 0.$$

In the next section it will be shown that conditions (2) to (5) uniquely determine L and further imply the concise representation

$$(6) \quad L_t = \sup_{0 \leq s \leq t} X_s^-.$$

Because X is defined in terms of primitive elements by (1), this completes the precise mathematical specification of our two-stage flow system model with infinite buffer capacity.

A critical feature of this construction is that L and Z depend on A and B only through their difference, so one may view X as the sole primitive element of our system model. Borrowing a term from the economic theory of production, we shall hereafter refer to X as a *netput process*. This same term will be used later in other contexts, always to describe a net of potential input less potential output. The development above requires that X have continuous sample paths, but thus far no probabilistic assumptions have been imposed. The emphasis in this chapter is on construction of sample paths rather than on probabilistic analysis.

§2. THE ONE-SIDED REGULATOR

Let $C \equiv C[0, \infty)$ as in §A.2. Elements of C will often be called paths or trajectories rather than functions, and the generic element of C will be denoted by $x = (x_t, t \geq 0)$. We now define mappings $\psi, \phi : C \rightarrow C$ by setting

$$(1) \quad \psi_t(x) \equiv \sup_{0 \leq s \leq t} x_s^- \quad \text{for } t \geq 0$$

and

$$(2) \quad \phi_t(x) \equiv x_t + \psi_t(x) \quad \text{for } t \geq 0.$$

For purposes of discussion, fix $x \in C$ and let $l \equiv \psi(x)$ and $z \equiv \phi(x) = x + l$. We shall say that z is obtained from x by imposition of a *lower control barrier* at zero. The mapping (ψ, ϕ) will be called the *one-sided regulator* with lower barrier at zero. The effect of this path-to-path transformation is shown graphically in Figure 2, where the dotted line is $-l_t$. Note that $l = 0$ and hence $z = x$ up until the first time t at which $x_t = 0$. Thereafter z_t equals the amount by which x_t exceeds the minimum value of x over $[0, t]$.

(3) Proposition. Suppose $x \in C$ and $x_0 \geq 0$. Then $\psi(x)$ is the unique function l such that

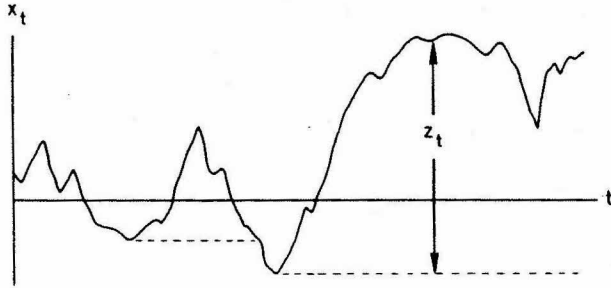


Figure 2. The one-sided regulator.

(4) l is continuous and increasing with $l_0 = 0$,

(5) $z_t \equiv x_t + l_t \geq 0$ for all $t \geq 0$, and

(6) l increases only when $z = 0$.

(7) Remark. Let l be any function on $[0, \infty)$ satisfying (4) and (5) alone. It is easy to show that $l_t \geq \psi_t(x)$ for all $t \geq 0$. In this sense, the *least* solution of (4) and (5) alone is obtained by taking $l = \psi(x)$.

Proof. Fix $x \in C$ and set $l \equiv \psi(x)$ and $z \equiv x + l$. It is left as an exercise to show that this l does in fact satisfy (4) to (6). To prove uniqueness, let l^* be any other solution of (4) to (6) and set $z^* \equiv x + l^*$. Setting $y \equiv z^* - z = l^* - l$, we note that y is a continuous VF function with $y_0 = 0$. Thus the Riemann–Stieltjes chain rule (B.4.1) gives

$$(8) \quad f(y_t) = f(0) + \int_0^t f'(y) dy$$

for any continuously differentiable $f: R \rightarrow R$. Taking $f(y) = y^2/2$, we see that (8) reduces to

$$(9) \quad \frac{1}{2} (z_t^* - z_t)^2 = \int_0^t (z^* - z) dl^* + \int_0^t (z - z^*) dl.$$

We know that l^* increases only when $z^* = 0$, and $z \geq 0$, so the first term on the right side of (9) is ≤ 0 , and identical reasoning shows that the second term is ≤ 0 as well. But because the left side is ≥ 0 , both sides must be zero. This shows that $z^* = z$ and hence $l^* = l$, and the proof is complete. $\square$

Note that the property $l_0 = 0$ in (4) depends critically on the assumption that $x_0 \geq 0$. The following proposition shows that our one-sided regulator

has a sort of *memoryless property*. It will be used later to prove the strong Markov property of regulated Brownian motion.

(10) Proposition. Fix $x \in C$ and set $l \equiv \psi(x)$ and $z \equiv \phi(x) = x + l$. Fix $T > 0$ and define $x_t^* = z_T + (x_{T+t} - x_T)$, $l_t^* = l_{T+t} - l_T$, and $z_t^* = z_{T+t}$ for $t \geq 0$. Then $l^* = \psi(x^*)$ and $z^* = \phi(x^*)$.

Because the proof of (10) is just a matter of verification, it is left as an exercise. Pursuant to the observation (7), it is often helpful to think of l_t as the cumulative amount of *control* exerted by an observer of the sample path x up to time t . This observer must increase l fast enough to keep $z \equiv x + l$ positive but wishes to exert as little control as possible subject to this constraint.

§3. FINITE BUFFER CAPACITY

Consider again the two-stage flow system of §1, assuming now that the buffer has finite capacity b . Except as noted below, the assumptions and notation of §1 remain in force. In particular, the system netput process is defined by $X_t \equiv X_0 + A_t - B_t$, and L_t denotes the amount of potential output lost up to time t due to emptiness of the buffer. In the current context one must interpret A as a *potential* input process; some of this potential input may be lost when the buffer is full. For reasons that will become clear in the next section, we denote by U_t the total amount of potential input lost up to time t . Thus actual input up to time t is $A_t - U_t$, and the inventory process Z is given by

$$(1) \quad \begin{aligned} Z_t &= X_0 + (A_t - U_t) - (B_t - L_t) \\ &= X_t + L_t - U_t. \end{aligned}$$

Now how are L and U to be defined in terms of the primitive model elements? Assuming that $X_0 \in [0, b]$, it is more or less obvious from the development in §1 and §2 that L and U should be uniquely determined by the following properties:

(2) L and U are continuous and increasing with $L_0 = U_0 = 0$,

(3) $Z_t \equiv (X_t + L_t - U_t) \in [0, b]$ for all $t \geq 0$, and

(4) L and U increase only when $Z = 0$ and $Z = b$, respectively.

In the next section it will be shown that (2) to (4) do in fact determine L and U uniquely, although they cannot be expressed in neat formulas like (1.6).

Again a crucial point is that the processes of interest depend on primitive model elements only through the netput process X .

It is important to realize that a finite buffer may represent either a physical restriction on storage space or a policy restriction that shuts off input when buffer stock reaches a certain level. In the context of production systems, input is almost always controllable, and it is simply irrational to let inventory levels fluctuate without restriction. Thus the model described here is fundamentally more interesting than that developed in §1 and will be the focus of attention later.

§4. THE TWO-SIDED REGULATOR

Fix $b > 0$ and let C^* be the set of all functions $x \in C$ such that $x_0 \in [0, b]$. Given $x \in C^*$, we would like to find a pair of functions (l, u) such that

- (1) l and u are increasing and continuous with $l_0 = u_0 = 0$,
- (2) $z_t \equiv (x_t + l_t - u_t) \in [0, b]$ for all $t \geq 0$, and
- (3) l and u increase only when $z = 0$ and $z = b$, respectively.

Note that (3) associates l and u with the *lower* barrier at zero and *upper* barrier at b , respectively. If we consider u to be given, then the requirements imposed on l by (1) to (3) are those that define a lower control barrier at zero. That is, (1) to (3) and Proposition (2.3) together imply that

$$(4) \quad l_t = \psi_t(x - u) \equiv \sup_{0 \leq s \leq t} (x_s - u_s)^-.$$

In exactly the same way, u may be expressed in terms of l via

$$(5) \quad u_t = \psi_t(b - x - l) \equiv \sup_{0 \leq s \leq t} (b - x_s - l_s)^-.$$

It will now be proved that (4) and (5) together uniquely determine l and u . The function z defined by (2) may be pictured as in Figure 3, where the lower dotted line is $u_t - l_t$ and the upper dotted line is $b + u_t - l_t$. We shall henceforth say that z is obtained from x through imposition of a *lower control barrier at zero and an upper control barrier at b* .

(6) Proposition. For each $x \in C^*$, there is a unique pair of continuous functions (l, u) satisfying (4) and (5), and this same pair uniquely satisfies (1) to (3).

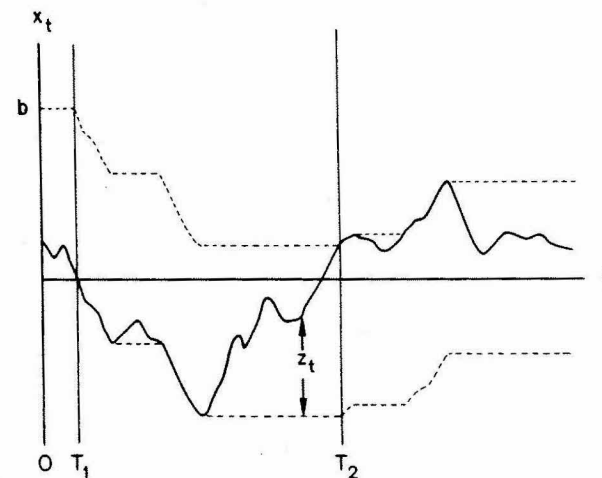


Figure 3. The two-sided regulator.

(7) Definition. We define mappings $f, g, h: C^* \rightarrow C$ by setting $f(x) \equiv l$, $g(x) \equiv u$, and $h(x) \equiv x + l - u$. Hereafter (f, g, h) will be called the *two-sided regulator* with lower barrier at zero and upper barrier at b .

Proof. We first construct a solution of (4) and (5) by successive approximations. Beginning with the trial solution $l_t^0 \equiv u_t^0 \equiv 0$ ($t \geq 0$), we set

$$(8) \quad l_t^{n+1} \equiv \psi_t(x - u^n) \equiv \sup_{0 \leq s \leq t} (x_s - u_s^n)^-$$

and

$$(9) \quad u_t^{n+1} \equiv \psi_t(b - x - l^n) \equiv \sup_{0 \leq s \leq t} (b - x_s - l_s^n)^-$$

for $n = 0, 1, \dots$ and $t \geq 0$. Observe that $l_t^1 \geq l_t^0$ and $u_t^1 \geq u_t^0$ for all t , and hence (by induction) that l_t^n and u_t^n are increasing in n for each fixed t . Thus we have

$$(10) \quad l_t^n \uparrow l_t \quad \text{and} \quad u_t^n \uparrow u_t \quad \text{as} \quad n \uparrow \infty.$$

Furthermore, it is easy to show that the convergence is achieved in a finite number of iterations for each fixed t , and the requisite number of iterations is an increasing function of t . For example, in Figure 3 we have $l_t = l_t^0$ and

$u_t = u_t^0$ if $0 \leq t \leq T_1$, $l_t = l_t^1$ and $u_t = u_t^1$ if $T_1 \leq t \leq T_2$, and so forth. (It is left as an exercise to show that $T_n \rightarrow \infty$, using the assumed continuity of x .) From this and (8) and (9) it follows that the limit functions l and u are finite valued, are continuous, and jointly satisfy (4) and (5).

To prove uniqueness, let (l, u) and (l^*, u^*) be two pairs of continuous functions satisfying (4) and (5), and let $z \equiv x + l - u$ and $z^* \equiv x + l^* - u^*$. From Proposition (2.3) it follows that (l, u) and (l^*, u^*) both satisfy (1) to (3) as well. Now let $y \equiv z^* - z = (l^* - l) - (u^* - u)$. Using the Riemann-Stieltjes chain rule as in the proof of Proposition (2.3), we find that

$$(11) \quad \frac{1}{2} (z_t^* - z_t)^2 = \int_0^t (z^* - z) dl + \int_0^t (z - z^*) dl^* \\ + \int_0^t (z - z^*) du + \int_0^t (z^* - z) du^*.$$

Also as in the proof of Proposition (2.3), we use (1) to (3) to conclude that each term on the right side of (11) is ≤ 0 , whereas the left side is ≥ 0 , and hence each side is zero. Thus $z^* = z$, from which it follows easily that $l^* = l$ and $u^* = u$ so that there is exactly one continuous pair (l, u) satisfying (4) and (5). As we observed earlier, (1) to (3) and (4) and (5) are equivalent for continuous pairs (l, u) by (2.3), and this proves the last statement of the proposition. $\square$

(12) Corollary. For each fixed t , both $l_t \equiv f_t(x)$ and $u_t \equiv g_t(x)$ depend on x only through $(x_s, 0 \leq s \leq t)$.

Proof. Immediate from the construction (8) to (10). $\square$

(13) Proposition. Fix $x \in C$ and let $l \equiv f(x)$, $u \equiv g(x)$, and $z \equiv h(x)$ as above. Fix $T > 0$ and define $x_t^* \equiv x_T + (x_{T+t} - x_T)$, $l_t^* \equiv l_{T+t} - l_t$, $u_t^* \equiv u_{T+t} - u_t$, and $z_t^* \equiv z_{T+t}$ for $t \geq 0$. Then $l^* = f(x^*)$, $u^* = g(x^*)$, and $z^* = h(x^*)$.

Proof. Starting with the fact that x, l, u, z all satisfy (1) to (3), it is easy to verify that x^*, l^*, u^*, z^* satisfy these same relations. The second uniqueness statement of (6) then establishes the desired proposition. $\square$

§5. MEASURING SYSTEM PERFORMANCE

In the design and operation of buffered flow systems, one is typically concerned with a tradeoff between system throughput characteristics and

the costs associated with inventory. Generally speaking, one can decrease the amount of lost potential input and output (which amounts to improving capacity utilization) by tolerating larger buffer stocks, but such stocks are costly in their own right.

To put the discussion on a concrete footing, consider again the single-product firm described at the beginning of this chapter. Recall that production flows into a finished goods inventory, and demand that cannot be met from stock on hand is simply lost with no adverse effect on future demand. Let π denote the selling price (in dollars per unit of production) and let B_t denote total demand over the time interval $[0, t]$. The latter notation is chosen for consistency with previous usage in §1 and §3.

Assuming plant and equipment are fixed, suppose that the firm must select at time zero a work force size, or equivalently a regular-time production capacity. For simplicity, assume that the work force size cannot be varied thereafter, the firm being obliged to pay workers their regular wages regardless of whether they are productively employed. Let k be the capacity level selected, in units of production per unit time. The firm then incurs a labor cost of wk dollars per unit time ever afterward, where $w > 0$ is a specified wage rate, even if it occasionally chooses to operate below capacity. For current purposes, overtime production is assumed to be impossible (see Problem 8). In addition to its labor costs, the firm incurs a materials cost of m dollars per unit of *actual* production. Given the initial capacity decision (work force level), labor costs are fixed, and thus the *marginal cost of production is m dollars per unit*. A physical holding cost of p dollars is incurred per unit time for each unit of production held in inventory. This includes such costs as insurance and security; it does *not* include the financial cost of holding inventory. (By financial cost we mean the opportunity loss on money tied up in inventory. More will be said on this subject shortly.)

It is assumed that the firm earns interest at rate $\lambda > 0$, compounded continuously, on funds that are not required for production operations. Continuous compounding means that one dollar invested at time zero returns $\exp(\lambda t)$ dollars of principal plus interest at time t . Thus a cost or revenue of one dollar at time t is equivalent in value to a cost or revenue of $\exp(-\lambda t)$ dollars at time zero. Finally, we assume that the cumulative demand process B satisfies

- (1) $E(B_t) = at$ for all $t \geq 0$ ($a > 0$) and
- (2) $e^{-\lambda t} B_t \rightarrow 0$ almost surely as $t \rightarrow \infty$.

For one specific demand model that satisfies (1) and (2), we may suppose that the time axis can be divided into periods of unit length, that demand

increments during successive periods form a sequence of independent and identically distributed random variables with mean a and finite variance, and that demand arrives at a constant rate during each period. For this *linearized random walk* model of demand, property (1) is obvious and (2) follows from the strong law of large numbers. (The proof of this statement is left as an exercise.)

The firm must choose a capacity level k at time zero and then at each time $t \geq 0$ select a production rate from the interval $[0, k]$. When a production rate below k is selected, we shall say that *undertime* is being employed. For purposes of initial discussion, let us assume that management follows a *single-barrier policy* for production control after time zero. This means that production continues at the capacity rate k until inventory hits some chosen level $b > 0$, and then undertime is employed in the minimum amounts necessary to keep inventory at or below level b . With this policy, our make-to-stock production system is a two-stage flow system with finite buffer capacity (see §3); the potential input process is $A_t \equiv kt$, and potential output is given by the demand process B . In the current context, Z_t represents the finished goods inventory level at time t , L_t is the cumulative demand lost up to time t , and U_t is the cumulative undertime worked (potential production foregone) up to time t .

The firm's objective is to maximize the expected present value of sales revenues received minus operating expenses incurred over an infinite planning horizon, where discounting is continuous at interest rate λ . The actual production and sales volumes up to time t are given by $kt - U_t$ and $B_t - L_t$, respectively; thus this amounts to maximization of

$$(3) \quad V \equiv E \left[\pi \int_0^\infty e^{-\lambda t} (dB - dL) - wk \int_0^\infty e^{-\lambda t} dt - m \int_0^\infty e^{-\lambda t} (k dt - dU) - p \int_0^\infty e^{-\lambda t} Z_t dt \right],$$

where the integrals involving dB , dL , and dU are defined path by path in the Riemann–Stieltjes sense (see Appendix B). The first term inside the expectation in (3) represents the present value of sales revenues, the second is the present value of labor costs, the third term is the present value of material costs (incremental production costs), and the last is the present value of inventory holding costs. It should be emphasized that the opportunity loss on capital tied up in inventory is fully accounted for by the discounting in (3); therefore p should include only out-of-pocket expenses associated with holding inventory. To put it another way, no explicit financial cost of holding

inventory appears in (3) and including such a cost would be double counting. In a moment, however, we shall derive an equivalent measure of system performance in which a financial cost of inventory *does* appear. Readers who are not familiar with present value manipulations, and skeptical as to the appropriateness of (3) as a performance measure, may wish to consult §6.5. There it is shown that maximization of a discounted measure like V is equivalent to maximizing the firm's expected total assets at a distant time of reckoning.

It will now be shown that maximization of V is equivalent to minimization of another, somewhat simpler, performance measure. As a first step, consider the ideal situation where $B_t = at$ for all $t \geq 0$, meaning that demand arrives deterministically at constant rate a . We shall assume that

$$(4) \quad \pi - w - m > 0,$$

for otherwise the system optimization problem would be uninteresting. (If $\pi - w - m \leq 0$, it is best to set $k = 0$ and go out of business.) With deterministic demand, one would, of course, choose $k = a$, meaning that units are produced precisely as demanded, labor and materials are paid for only as required for such production, and no inventory is held. The corresponding *ideal profit level* (in present value terms) would be

$$(5) \quad I \equiv \int_0^\infty e^{-\lambda t} (\pi - w - m)a dt = \frac{(\pi - w - m)a}{\lambda}.$$

Now actual system performance under an arbitrary operating policy will be measured incrementally from this ideal. First, let

$$(6) \quad \mu \equiv k - a,$$

$$(7) \quad \delta \equiv \pi - m, \quad \text{and} \quad h \equiv p + m\lambda.$$

We call μ the *excess capacity*; it is the amount (possibly negative) by which chosen capacity exceeds the average demand rate. Interpret δ as a *contribution margin*; once the capacity level is fixed, each unit of sales contributes δ dollars to profit and the coverage of fixed costs. Finally, h may be viewed as the *effective cost of holding inventory*; it consists of physical holding costs plus an opportunity loss rate of λ times the marginal production cost m . It is assumed hereafter that $Z_0 = 0$.

(8) Proposition. $V = I - \Delta$, where

$$(9) \quad \Delta \equiv E \left[\int_0^\infty e^{-\lambda t} (\delta dL + w\mu dt + hZ_t dt) \right].$$

(10) Remark. Because demand is exogenous, I is an uncontrollable constant, and thus our original objective of maximizing V is equivalent to minimizing Δ .

Proof. From (1) and (2) it follows that

$$(11) \quad E \left(\int_0^\infty e^{-\lambda t} dB \right) = E \left(\int_0^\infty \lambda e^{-\lambda t} B_t dt \right) \\ = \int_0^\infty \lambda e^{-\lambda t} at dt = \int_0^\infty e^{-\lambda t} a dt.$$

The proof of (11), using Fubini's theorem and the Riemann–Stieltjes integration by parts theorem, is left as an exercise. Using (11), we can rewrite (5) as

$$(12) \quad I = E \left[\int_0^\infty e^{-\lambda t} (\pi dB - wa dt - m dB) \right].$$

Now subtracting (3) from (12) we get

$$(13) \quad I - V = E \left\{ \int_0^\infty e^{-\lambda t} [\pi dL + w(k - a) dt + pZ_t dt + m(k dt - dU - dB)] \right\}.$$

With $Z_0 = 0$, we have $Z_t = (kt - U_t) - (B_t - L_t)$. Using this and integration by parts again, we find that

$$(14) \quad \int_0^\infty e^{-\lambda t} (k dt - dU - dB) = \int_0^\infty e^{-\lambda t} (dZ - dL) \\ = \int_0^\infty e^{-\lambda t} (\lambda Z_t dt - dL).$$

Substituting (14) into (13) and collecting similar terms, we have $I - V = \Delta$. $\square$

Obviously Δ represents the amount by which management's plan falls short, in expected present value terms, of the ideal profit level I . The definition (9) expresses this shortfall as the sum of three effects. First, the contribution margin δ is lost on each unit of potential sales foregone. Second, we continuously incur a cost of w dollars for each unit of capacity in excess of the average demand rate. Finally, for each unit of production held in inventory, we continuously incur an out-of-pocket cost p plus an opportunity cost λm . We emphasize again that Δ measures the *degradation of system performance from a deterministic ideal*. Thus the minimum achievable Δ value may be viewed as the *cost of stochastic variability*.

Our first objective here is to develop a quantitative theory of flow system performance. As a natural outgrowth of that descriptive objective, we also seek to prescribe means by which management can minimize or at least reduce performance degradation, such as investment in excess capacity (a design decision) and maintenance of buffer stock (a matter of operating policy).

In concluding this section, let us briefly consider a cost structure in which $\lambda \downarrow 0$ but δ , w , and h remain constant. Further suppose that

$$(15) \quad \frac{1}{t} E(L_t) \rightarrow \alpha \text{ and } E(Z_t) \rightarrow \gamma \text{ as } t \rightarrow \infty.$$

Obviously α represents a long-run average *lost sales rate*, whereas γ is the long-run average inventory level. Under mild additional assumptions, it is well known that $\lambda\Delta$ approaches the *long-run average cost rate*

$$(16) \quad \rho \equiv \delta\alpha + w\mu + h\gamma$$

as $\lambda \downarrow 0$. Thus minimization of Δ is approximately equivalent to minimization of ρ for small values of λ , and it is usually easier to calculate ρ than the discounted performance measure Δ .

§6. BROWNIAN FLOW SYSTEMS

Suppose that, in the setting of §3, we directly model the netput process X as a (μ, σ) Brownian motion. The inventory process Z , lost potential output L , and lost potential input U are then defined by applying the two-sided regulator to X exactly as before. In the obvious way, we call Z a *regulated Brownian motion*, and the triple (L, U, Z) will be referred to hereafter as a *Brownian flow system*. It will be seen later that all the performance measures

discussed in §5, and a number of other interesting quantities, can be calculated explicitly for Brownian flow systems.

Although the Brownian system model is tractable, and therefore appealing, it is actually inconsistent with the model description given in §3; we have seen earlier that the sample paths of Brownian motion have infinite variation and thus it cannot represent the difference between a potential input process and a potential output process. Nonetheless, a netput process may be well approximated by Brownian motion under certain conditions. To understand these conditions, recall that Brownian motion is the *unique* stochastic process having stationary, independent increments and continuous sample paths; unbounded variation follows as a consequence of these primitive properties. Also note that the total variation of a netput process over any given interval equals the *sum* of potential input and potential output over that interval. If such a netput process is to be well approximated by Brownian motion, both potential input and potential output must be large for intervals of moderate length, but their difference (netput itself) must be moderate in value. We may express this state of affairs by saying that we have a system of *balanced high-volume flows*.

Pulling together several times, we conclude that Brownian motion may reasonably approximate the netput process for a system of *stationary, continuous, balanced high-volume flow, where netput increments during non-overlapping intervals are approximately independent*. Formal limit theorems that give this statement precise mathematical form, and thus serve to justify Brownian approximations, have been proved for various types of flow system models. The Brownian flow system will be studied extensively in future chapters, and readers should keep in mind its domain of applicability.

PROBLEMS AND COMPLEMENTS

1. Prove Proposition (2.10), thus verifying the one-sided regulator's lack of memory.
2. Prove that $l \equiv \psi(x)$ satisfies (2.4) to (2.6).
3. Consider the three-stage flow system, or tandem buffer system, pictured in Figure 4. Each buffer has infinite capacity, and we denote by $X_k(0)$ the initial inventory in buffer k . Extending in an obvious way the model of §1, we take as primitive three increasing, continuous processes $A_k = \{A_k(t), t \geq 0\}$ such that $A_k(0) = 0$ ($k = 1, 2, 3$). Interpret A_1 as input to the first buffer, A_2 as potential transfer between the two buffers, and A_3 as potential output from the second buffer. Define a (continuous) vector netput process $X(t) = [X_1(t), X_2(t)]$ by setting



Figure 4. A three-stage flow system.

$$X_1(t) = X_1(0) + A_1(t) - A_2(t) \quad \text{for } t \geq 0$$

and

$$X_2(t) = X_2(0) + A_2(t) - A_3(t) \quad \text{for } t \geq 0.$$

Let $L_2(t)$ denote the amount of the potential transfer $A_2(t)$ that is lost over $[0, t]$ because of emptiness of the first buffer, and define $L_3(t)$ in the obvious analogous fashion. Let $Z_k(t)$ denote the content of buffer k at time t . Applying the analysis of §1 and §2 first to buffer 1 and then to buffer 2 in isolation, show that $L_2 = \psi(X_1)$, $Z_1 = \phi(X_1)$, $L_3 = \psi(X_2 - L_2)$, and $Z_2 = \phi(X_2 - L_2)$. Conclude that $L \equiv (L_2, L_3)$ and $Z \equiv (Z_1, Z_2)$ uniquely satisfy

- (a) L_2 and L_3 are increasing and continuous with $L_2(0) = L_3(0) = 0$.
- (b) $Z_1(t) = X_1(t) + L_2(t) \geq 0$ for all $t \geq 0$,
 $Z_2(t) = X_2(t) - L_2(t) + L_3(t) \geq 0$ for all $t \geq 0$.
- (c) L_2 and L_3 increase only when $Z_1 = 0$ and $Z_2 = 0$, respectively.

All of this describes the mapping by which (L, Z) is obtained from X . (It is again important that L and Z depend on primitive model elements only through the netput process X .) Conditions (a) to (c) suggest the following interpretation or animation of that path-to-path transformation. An observer watches $X = (X_1, X_2)$ and may increase at will either component of a cumulative control process $L = (L_2, L_3)$. These actions determine $Z = (Z_1, Z_2)$ according to (b). The observer increases L_2 only as necessary to ensure that $Z_1 \geq 0$, so L_2 increases only when $Z_1 = 0$. Each such increase causes a positive displacement of Z_1 (or rather prevents a negative one) and an equal negative displacement of Z_2 . Thus the effect of the observer's action at $Z_1 = 0$ is to drive Z in the diagonal direction pictured in Figure 5. On the other hand, L_3 is increased at the boundary $Z_2 = 0$ so as to ensure $Z_2 \geq 0$, producing only the vertical displacement picture in Figure 5. Hereafter we shall say that (L, Z) is obtained by applying a *multidimensional regulator* to X , the control region and *directions of control* being as illustrated in Figure 5. This problem is adapted from Harrison (1978).

4. A similar sort of multidimensional flow system is pictured in Figure 6. Here there are two input processes, each feeding its own infinite storage buffer. These inputs are then combined, exactly one unit of each input being required to produce one unit of system output. (The important

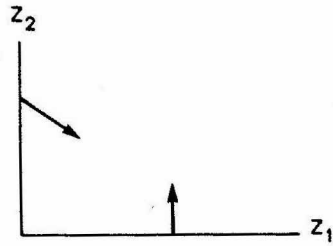


Figure 5. Directions of control for a three-stage flow system.

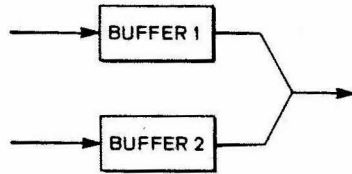


Figure 6. An assembly or blending operation.

point here is that inputs are combined in *fixed proportions*; the rest is just a matter of how units are defined.) This is the structure of an assembly operation, but again we treat the system flows as if they were continuous so that attention is effectively restricted to high-volume assembly systems. For another application, Figure 6 might be interpreted as a blending operation in which liquid or granulated ingredients are combined in fixed proportions to produce a similarly continuous output. To build a model, we again take as primitive initial inventory levels $X_1(0) \geq 0$ and $X_2(0) \geq 0$ plus three increasing, continuous processes $A_k = \{A_k(t), t \geq 0\}$ with $A_k(0) = 0$ ($k = 1, 2, 3$). Interpret A_1 and A_2 as input to buffer 1 and buffer 2, respectively, and A_3 as potential output. Potential output is lost if *either* buffer is empty, and we denote by $L(t)$ the cumulative potential output lost up to time t because of such emptiness. For purposes of determining L , the blending operation may be viewed as a two-stage flow system with initial inventory plus cumulative input given by

$$A^*(t) = [X_1(0) + A_1(t)] \wedge [X_2(0) + A_2(t)].$$

Let $Z_k(t)$ denote the inventory level in buffer k at time t , and define a (continuous) vector netput process $X = [X_1(t), X_2(t)]$ by setting

$$X_1(t) \equiv X_1(0) + A_1(t) - A_3(t) \quad \text{for } t \geq 0$$

and

$$X_2(t) \equiv X_2(0) + A_2(t) - A_3(t) \quad \text{for } t \geq 0.$$

Applying the results of §1 and §2, write out explicit formulas for L and $Z \equiv (Z_1, Z_2)$ in terms of X . (Again it is important that L and Z depend on primitive model elements only through the netput process X .) Conclude that L and Z together uniquely satisfy

- (a) L is continuous and increasing with $L(0) = 0$.
- (b) $Z_1(t) = X_1(t) + L(t) \geq 0$ for all $t \geq 0$,
 $Z_2(t) = X_2(t) + L(t) \geq 0$ for all $t \geq 0$.
- (c) L increases only when $Z_1 = 0$ or $Z_2 = 0$.

The mapping that carries X into (L, Z) may be pictured as in Figure 7. The inventory process Z coincides with X up until X hits the boundary of the positive quadrant. At that point, L increases, causing *equal positive displacements in both Z_1 and Z_2* as necessary to keep $Z_1 \geq 0$ and $Z_2 \geq 0$. Thus the effect of increases in L at the boundary is to drive Z in the diagonal direction shown in Figure 7, regardless of which boundary surface is struck. This problem is adapted from Harrison (1973).

- 5. Assuming for convenience that $x_0 = 0$, write out an explicit recursive expression for the times $T_1 < T_2 < \dots$ identified in the proof of Proposition (4.6). Show that if $T_n \uparrow T < \infty$, then x cannot be continuous at T ; thus $T_n \rightarrow \infty$ as $n \rightarrow \infty$.
- 6. Consider again the three-stage flow system of Problem 3, assuming that buffers 1 and 2 now have *finite* capacities b_1 and b_2 , respectively. In this case, potential input is lost when the first buffer is full, and potential transfer is lost when *either* the first buffer is empty *or* the second one is full. (We say that the transfer process is *starved* in the former case and *blocked* in the latter.) In addition to the notation established in Problem 3, let $L_1(t)$ denote total potential input lost up to time t . Argue that $L \equiv (L_1, L_2, L_3)$ and $Z \equiv (Z_1, Z_2, Z_3)$ should jointly satisfy
 - (a) L_k is continuous and increasing with $L_k(0) = 0$ ($k = 1, 2, 3$).
 - (b) $Z_1(t) \equiv X_1(t) + L_2(t) - L_1(t) \in [0, b_1]$ for all $t \geq 0$,
 $Z_2(t) \equiv X_2(t) + L_3(t) - L_2(t) \in [0, b_2]$ for all $t \geq 0$.

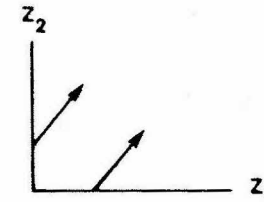


Figure 7. Directions of control for a blending operation.

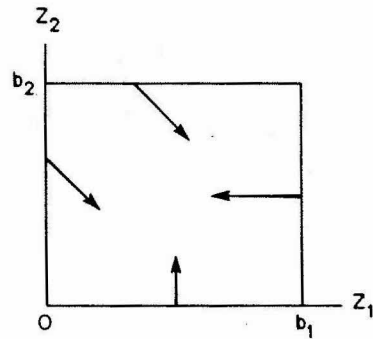


Figure 8. Directions of control for a three-stage flow system with finite buffers.

- (c) L_1 increases only when $Z_1 = b_1$,
 L_2 increases only when $Z_1 = 0$ or $Z_2 = b_2$, and
 L_3 increases only when $Z_2 = 0$.

Explain the connection between (a) to (c) and Figure 8. Describe informally how one can use the results of Problem 3 and 4 to prove existence and uniqueness of a pair (L, Z) satisfying (a) to (c). This problem is adapted from Wenocur (1982).

7. Show that the linearized random walk model of demand, described in §5, satisfies (5.1) and (5.2).
8. It was assumed in §5 that overtime production was impossible. Suppose instead that unlimited amounts of overtime production are available at a premium wage rate $w^* > w$, regardless of what work force level may be chosen at the beginning. To keep things simple, assume that overtime production is instantaneous. (One may also think in terms of buying finished goods at a premium price from some alternate supplier and then using these goods to satisfy demand.) Finally, assume that $\pi - w^* - m > 0$, so it is always better to use overtime production than to forego potential sales. The basic structure of this system is identical to that discussed in §5, but now L_t is interpreted as cumulative overtime production up to time t . Show that maximizing the expected present value of total profit is equivalent to minimizing Δ , where Δ is given by formula (5.9) with w^* in place of δ .
9. Prove the three equalities of (5.11), using Fubini's theorem (§A.5) and the Riemann–Stieltjes integration by parts theorem (§B.3).

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Further Analysis of Brownian Motion

The treatment of Brownian motion in Chapter 1 was restricted to the case $X_0 = 0$. As we move on to more complex calculations, it will be convenient to view the starting state as a variable parameter. This is accomplished by introducing a family of probability measures on path space, with each member of the family corresponding to a different starting state.

§0. INTRODUCTION

Throughout this chapter let $(\Omega, \mathcal{F}) = (C, \mathcal{C})$ and let X be the coordinate process on Ω as in §A.3. Let μ and $\sigma > 0$ be fixed constants. For each $x \in R$ there is a unique probability measure P_x on $(\Omega, \mathcal{F})$ such that

$$(1) \quad X \text{ is a } (\mu, \sigma) \text{ Brownian motion on } (\Omega, \mathcal{F}, P_x)$$

and

$$(2) \quad P_x\{\omega \in \Omega : X_0(\omega) = x\} = 1.$$

This follows from Wiener's theorem (1.1.1). We paraphrase (1) and (2) by saying that X is a (μ, σ) Brownian motion *with starting state x* under P_x . Heuristically, one may think of $P_x(A)$ as the conditional probability of event A given that $X_0 = x$. Let E_x be the expectation operator associated with P_x . That is,

$$E_x(Z) \equiv \int_{\Omega} Z(\omega) P_x(d\omega)$$

for all measurable functions (random variables) $Z: \Omega \rightarrow R$ such that the integral on the right exists. Finally, let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the filtration generated by X (see §A.3) throughout this chapter. This filtration is implicitly referred to whenever we speak of stopping times and martingales.

With this setup, the strong Markov property of X can be recast in the following form. Let T be an arbitrary stopping time and set

$$(3) \quad X_t^* \equiv X_{T+t}, \quad t \geq 0, \quad \text{on } \{T < \infty\}.$$

(This is *not*, however, the definition of X^* that was used in §1.4.) More precisely, (3) means that

$$X_t^*(\omega) \equiv X_{T(\omega)+t}(\omega), \quad t \geq 0,$$

for ω such that $T(\omega) < \infty$. The process X^* need not be defined at all on $\{\omega : T(\omega) = \infty\}$. Now let F be a measurable mapping $(C, \mathcal{C}) \rightarrow (R, \mathcal{B})$ such that $E_x\{|F(x)|\} < \infty$ for all $x \in R$ and define

$$(4) \quad f(x) \equiv E_x[F(X)], \quad x \in R.$$

From our original articulation (1.4.1) of the strong Markov property it follows that

$$(5) \quad E_x[F(X^*)|\mathcal{F}_T] = f(X_0^*) \equiv f(X_T) \text{ on } \{T < \infty\}.$$

Readers will find that (5) plays a major role in §4 and §5, where we calculate expected discounted costs for Brownian motion with absorbing barriers.

§1. THE BACKWARD AND FORWARD EQUATIONS

Recall that $X_{t+s} - X_t \sim N(\mu s, \sigma^2 s)$ under P_x for each $x \in R$. Thus the transition density

$$p(t, x, y) dy \equiv P_x\{X_t \in dy\} \quad \text{for } t \geq 0 \text{ and } x, y \in R$$

is given by

$$(1) \quad p(t, x, y) = (\sigma^2 t)^{-1/2} \phi\left(\frac{y - x - \mu t}{\sigma t^{1/2}}\right),$$

where $\phi(z) \equiv (2\pi)^{-1/2} \exp(-z^2/2)$ is the standard normal density function. Direct calculation shows (see Problem 1) that p satisfies

$$(2) \quad \frac{\partial}{\partial t} p(t, x, y) = \left(\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) p(t, x, y)$$

with initial condition

$$(3) \quad p(0, x, y) = \delta(x - y).$$

Here $\delta(\cdot)$ is the Dirac delta function; (3) is defined to mean that

$$\int_R h(y) p(t, x, y) dy \rightarrow h(x) \text{ as } t \downarrow 0$$

for all bounded, continuous $h: R \rightarrow R$. This differential equation for the transition density p will not play much of a role in our future analysis of Brownian motion. It has, however, played a major role in the historical development of the subject, and a little more commentary is appropriate. In probability theory, (2) is called *Kolmogorov's backward equation* for the Markov process X . Computing directly from (2), readers may verify the corresponding *forward equation*

$$(4) \quad \frac{\partial}{\partial t} p(t, x, y) = \left(\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial y^2} - \mu \frac{\partial}{\partial y} \right) p(t, x, y).$$

Note that we differentiate with respect to the *backward variable* (initial state) x on the right side of (2), whereas (4) involves differentiation with respect to the *forward variable* (final state) y . In the special case where $\mu = 0$, equation (4) reduces to the celebrated *heat equation* (or diffusion equation) of mathematical physics. Because of this connection with the mathematics of physical diffusion, Brownian motion and certain of its close relatives are called *diffusion processes*. One could hardly find a worse name to describe the sample path behavior of these processes.

§2. HITTING TIME PROBLEMS

Hereafter let $T(y)$ denote the first time $t \geq 0$ at which $X_t = y$, with $T(y) = \infty$ if no such t exists. Fixing $b > 0$, we restrict attention to starting states $X_0 = x \in [0, b]$ and let $T \equiv T(0) \wedge T(b)$ as in Figure 1. The ob-

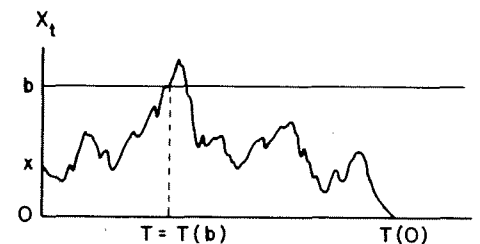


Figure 1. First exit time T from $[0, b]$.

jective in this section is to calculate the Laplace transform $E_x[\exp(-\lambda T)]$ and various other related quantities. In particular, we shall compute the Laplace transforms of $T(0)$ and $T(b)$ and the probability that level b is hit before level zero. It will be seen later that the formulas derived in this section are indispensable for computing expected discounted costs.

(1) Proposition. $E_x(T) < \infty$, $0 \leq x \leq b$.

Proof. First consider the case $\mu > 0$. Let $M_t \equiv X_t - \mu t$ for $t \geq 0$. A slight modification of the argument given in §1.5 shows that M is a martingale on $(\Omega, \mathbb{F}, P_x)$ and thus the martingale stopping theorem (see §A.4) says that $E_x[M(T \wedge t)] = E_x(M_0)$. That is,

$$(2) \quad E_x[X(T \wedge t)] - \mu E_x(T \wedge t) = x.$$

Of course, $X(T \wedge t) \leq b$, so $E_x(T \wedge t) \leq (b - x)/\mu$ by (2). Because this holds for any $t > 0$, we have $E_x(T) \leq (b - x)/\mu < \infty$. The case $\mu < 0$ is handled symmetrically. Finally, if $\mu = 0$, it follows from (1.5.6) that $\{X_t^2 - \sigma^2 t, t \geq 0\}$ is a martingale on $(\Omega, \mathbb{F}, P_x)$. Then the martingale stopping theorem gives

$$(3) \quad E_x[X^2(T \wedge t) - \sigma^2(T \wedge t)] = E_x(X_0^2) = x^2$$

for any $t > 0$. But $X^2(T \wedge t) \leq b^2$ and therefore (3) gives $E_x(T \wedge t) \leq (b^2 - x^2)/\sigma^2$ for all $t > 0$, thus implying $E_x(T) \leq (b^2 - x^2)/\sigma^2 < \infty$. $\square$

Recall that in §1.5 we defined the Wald martingale V_β with dummy variable $\beta \in R$ via

$$(4) \quad V(t) \equiv \exp[\beta X_t - q(\beta)t], \quad t \geq 0,$$

where

$$(5) \quad q(\beta) \equiv \mu\beta + \frac{\sigma^2\beta^2}{2}.$$

(Again the argument given in §1.5 must be modified slightly to show that V_β is a martingale in our current setting, but readers should have no trouble supplying the details.) Hereafter we restrict attention to β values such that $q(\beta) \geq 0$. Then $\{V_\beta(T \wedge t), t \geq 0\}$ is a *bounded* family of random variables, and Corollary (A.4.2) of the martingale stopping theorem gives

$$(6) \quad E_x[V_\beta(T)] = E_x[V_\beta(0)] = e^{\beta x}, \quad 0 \leq x \leq b.$$

To further develop this identity, we introduce the notational convention

$$(7) \quad E_x(Z; A) \equiv \int_A Z dP_x$$

for events $A \in \mathcal{F}$ and random variables Z such that the integral on the right exists. Note that

$$(8) \quad E_x(Z; A) = E_x(Z|A) P_x(A).$$

Because it follows from (1) that $P_x\{T < \infty\} = 1$, Ω can be partitioned into the events $\{T = T(0) < \infty\}$ and $\{T = T(b) < \infty\}$. Then (6) gives us

$$(9) \quad e^{\beta x} = E_x[V_\beta(T); X_T = 0] + E_x[V_\beta(T); X_T = b] \\ = E_x[e^{-q(\beta)T}; X_T = 0] + E_x[e^{\beta b - q(\beta)T}; X_T = b].$$

To repeat, (9) holds for all $x \in [0, b]$ and all β such that $q(\beta) \geq 0$. Now for $\lambda > 0$ and $0 \leq x \leq b$ let us define

$$(10) \quad \psi_*(x|\lambda) \equiv E_x(e^{-\lambda T}; X_T = 0)$$

and

$$(11) \quad \psi^*(x|\lambda) \equiv E_x(e^{-\lambda T}; X_T = b).$$

Note that (9) can be reexpressed as

$$(12) \quad e^{\beta x} = \psi_*(x|q(\beta)) + e^{\beta b} \psi^*(x|q(\beta)).$$

Equation (12) will be used shortly to compute ψ_* and ψ^* . The Laplace transform of T (with dummy variable λ) is given by

$$(13) \quad E_x(e^{-\lambda T}) = \psi_*(x|\lambda) + \psi^*(x|\lambda), \quad 0 \leq x \leq b,$$

but one needs to know the terms ψ_* and ψ^* individually to compute expected discounted costs (see §4 and §5). From the definition (5) of $q(\cdot)$ we see that the *two* values of β that yield $q(\beta) = \lambda > 0$ are $\beta = \alpha^*(\lambda)$ and $\beta = -\alpha_*(\lambda)$, where

$$(14) \quad \alpha_*(\lambda) \equiv (1/\sigma^2)[(\mu^2 + 2\sigma^2\lambda)^{1/2} + \mu] > 0$$

and

$$(15) \quad \alpha^*(\lambda) \equiv (1/\sigma^2)[(\mu^2 + 2\sigma^2\lambda)^{1/2} - \mu] > 0.$$

These two roots are pictured in Figure 2 for a case where $\mu > 0$. (Note that $q'(0) = \mu$.) Substitution of $\beta = -\alpha_*(\lambda)$ and $\beta = \alpha^*(\lambda)$ into (12) gives

$$(16) \quad e^{-\alpha_*(\lambda)x} = \psi_*(x|\lambda) + e^{-\alpha_*(\lambda)b} \psi^*(x|\lambda)$$

and

$$(17) \quad e^{\alpha^*(\lambda)x} = \psi_*(x|\lambda) + e^{\alpha^*(\lambda)b} \psi^*(x|\lambda).$$

Hereafter we suppress the dependence of ψ_* and ψ^* on λ . Solving (16) and (17) simultaneously gives the following.

(18) Proposition. Let $\lambda > 0$ be fixed. For $0 \leq x \leq b$,

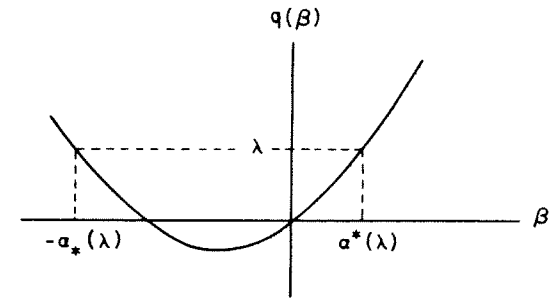


Figure 2. The two roots of $q(\beta) = \lambda > 0$.

$$(19) \quad \psi^*(x) = \frac{\theta^*(x) - \theta_*(x)\theta^*(0)}{1 - \theta_*(b)\theta^*(0)}$$

and

$$(20) \quad \psi_*(x) = \frac{\theta_*(x) - \theta^*(x)\theta_*(b)}{1 - \theta_*(b)\theta^*(0)},$$

where

$$(21) \quad \theta_*(x) \equiv \exp\{-\alpha_*(\lambda)x\}$$

and

$$(22) \quad \theta^*(x) \equiv \exp\{-\alpha^*(\lambda)(b-x)\}.$$

From the basic formulas (19) and (20) a variety of useful corollaries can be extracted. In the development to follow, let us agree to write

$$E_x[e^{-\lambda T(y)}] \equiv E_x[e^{-\lambda T(y)}; T(y) < \infty],$$

for $\lambda > 0$ and $x, y \in R$.

(23) Proposition. Let θ_* and θ^* be defined by (21) and (22) respectively. Then

$$(24) \quad E_x[e^{-\lambda T(0)}] = \theta_*(x), \quad 0 \leq x \leq b$$

and

$$(25) \quad E_x[e^{-\lambda T(b)}] = \theta^*(x), \quad 0 \leq x \leq b.$$

Proof. Let x be fixed. It can be shown that $T(b) \uparrow \infty$ as $b \uparrow \infty$, implying that $T \uparrow T(0)$ as $b \uparrow \infty$. Then the monotone convergence theorem gives

$$(26) \quad \psi_*(x) \equiv E_x(e^{-\lambda T}) \downarrow E_x[e^{-\lambda T(0)}] \text{ as } b \uparrow \infty.$$

From (20) and (22) we see that $\theta^*(x) \rightarrow 0$ as $b \rightarrow \infty$; hence $\psi_*(x) \rightarrow \theta_*(x)$. Combining this with (26) proves (24), and (25) is obtained symmetrically. $\square$

(27) Proposition. If $\mu = 0$, then $P_x\{X_T = b\} = x/b$, $0 \leq x \leq b$. Otherwise,

$$(28) \quad P_x\{X_T = b\} = \frac{1 - \xi(x)}{1 - \xi(b)}, \quad 0 \leq x \leq b,$$

where

$$(29) \quad \xi(z) \equiv \exp\left(\frac{-2\mu z}{\sigma^2}\right).$$

(30) Corollary. If $\mu \leq 0$, then $P_x\{T(0) < \infty\} = 1$ for all $x \geq 0$. If $\mu > 0$, then $P_x\{T(0) < \infty\} = \xi(x)$.

Proof. The monotone convergence theorem gives

$$\psi^*(x) \equiv E_x(e^{-\lambda T}; X_T = b) \uparrow P_x\{X_T = b\} \text{ as } \lambda \downarrow 0.$$

Proposition (27) follows immediately from this and the formula (19) for ψ^* . The corollary is then obtained by letting $b \uparrow \infty$ in (27). Alternatively, one can prove the corollary by letting $\lambda \downarrow 0$ in the formula developed earlier for $E_x[\exp\{-\lambda T(0)\}]$. $\square$

In the interest of efficiency, we have deduced (23), (27), and (30) from the master transform relation (18). See §7.5 of Karlin–Taylor (1976) and §13.7 of Breiman (1968) for different approaches to some of these results. In particular, Karlin–Taylor shows how we can obtain (23), (27), and (30) more directly, also using the Wald martingale V_β and the martingale stopping theorem.

For future reference, it will be useful to observe that each of the transforms computed in this section, viewed as a function of the starting state x , satisfies a characteristic second-order differential equation subject to particular boundary conditions. Specifically, if we define the differential operator

$$(31) \quad \Gamma \equiv \frac{1}{2} \sigma^2 \frac{d^2}{dx^2} + \mu \frac{d}{dx},$$

then it can be verified from the explicit formulas displayed above that (for $\lambda > 0$)

$$(32) \quad \lambda \theta_* - \Gamma \theta_* = \lambda \theta^* - \Gamma \theta^* = 0 \text{ on } R,$$

$$(33) \quad \theta_*(0) = \theta^*(b) = 1, \text{ and } \theta_*(\infty) = \theta^*(-\infty) = 0.$$

Consequently,

$$(34) \quad \lambda \psi_* - \Gamma \psi_* = \lambda \psi^* - \Gamma \psi^* = 0 \text{ on } (0, b),$$

$$(35) \quad \psi_*(0) = \psi^*(b) = 1 \text{ and } \psi_*(b) = \psi^*(0) = 0.$$

We have solved these simple ordinary differential equations by probabilistic methods, and more particularly by manipulation of the Wald martingale for Brownian motion. The relationship between Brownian motion and various differential equations will be developed further in the problems at the end of this chapter and in later chapters.

§3. EXPECTED DISCOUNTED COSTS

Let $\lambda > 0$ be fixed, and let u be a continuous function on R such that $|u|$ is bounded by a polynomial. It follows that

$$(1) \quad \int_0^\infty e^{-\lambda t} E_x\{|u(X_t)|\} dt < \infty \quad \text{for all } x \in R.$$

Now define

$$(2) \quad f(x) \equiv E_x \left[\int_0^\infty e^{-\lambda t} u(X_t) dt \right] = \int_0^\infty e^{-\lambda t} E_x[u(X_t)] dt, \quad x \in R.$$

The second equality in (2) follows from Fubini's theorem (§A.5). We interpret $u(y)$ as the rate at which costs are incurred when $X(t) = y$ and λ as the *interest rate* appropriate for discounting (see §2.5); thus $f(x)$ represents the *expected discounted cost* incurred over an infinite horizon, starting from level x . For certain specific functions u , one can explicitly calculate $E[u(X_t)]$ for general t and then perform the integration in (2). For example, if $u(x) = x$, we have $E_x[u(X_t)] = x + \mu t$, and a simple integration gives the following.

(3) Proposition. If $u(x) = x$, then $f(x) = x/\lambda + \mu/\lambda^2$.

An equally simple formula for $f(x)$ can be obtained in this way if the cost function $u(\cdot)$ is quadratic (see Problem 4). The next proposition gives a general formula for $f(x)$ as the integral of $u(\cdot)$ against a certain kernel.

(4) Proposition. $f(x) = \int_{-\infty}^\infty u(y) \pi(x, y) dy$, $x \in R$, where

$$(5) \quad \pi(x, y) = (\mu^2 + 2\lambda\sigma^2)^{-1/2} \theta(x, y)$$

and

$$(6) \quad \theta(x, y) = \begin{cases} \exp[-\alpha_*(\lambda)(x - y)] & \text{if } x \geq y \\ \exp[-\alpha^*(\lambda)(y - x)] & \text{if } y \geq x. \end{cases}$$

Because this general formula is not used later, we shall merely sketch its proof in Problems 5 to 8, where an interpretation for $\pi(x, y)$ is also given in terms of Brownian local time. To display the differential equation satisfied by the kernel π , let us fix y and agree to write $\pi(x) \equiv \pi(x, y)$. Readers may verify that π is continuous, is twice continuously differentiable except at $x = y$, and satisfies

$$(7) \quad \Gamma \pi(x) - \lambda \pi(x) = 0 \quad \text{except at } x = y$$

and

$$(8) \quad \frac{1}{2} \sigma^2 \Delta \pi'(y) = -1,$$

where Γ is the differential operator defined by (2.31) and $\Delta \pi'(y)$ is the jump in the derivative of $\pi(\cdot)$ going from left to right through $x = y$. It is left as an exercise to show, using (4), (7) and (8), that

$$(9) \quad \lambda f(x) - \Gamma f(x) = u(x) \quad \text{for all } x \in R.$$

This relationship between the cost function u and the expected discounted cost f will be studied further in Problems 4.2 and 4.3.

§4. ONE ABSORBING BARRIER

Now we restrict our attention to positive starting states x and set $T \equiv T(0)$ throughout this section, and define $Y_t \equiv X(T \wedge t)$, $t \geq 0$. Thus Y is (μ, σ)

Brownian motion with starting state x and *absorption at the origin* under P_x . The first goal of this section is to compute (for $x, y \geq 0$)

$$(1) \quad G(t, x, y) \equiv P_x\{Y_t > y\} = P_x\{X_t > y; T(0) > t\}.$$

Recall that in §1.6 we derived the joint distribution of X_t and M_t in the case $X_0 = 0$, where $M_t \equiv \sup\{X_s, 0 \leq s \leq t\}$. From this we can deduce that

$$(2) \quad G(t, x, y) = \Phi\left(\frac{-y + x + \mu t}{\sigma t^{1/2}}\right) - \exp\left(\frac{-2\mu x}{\sigma^2}\right) \Phi\left(\frac{-y - x + \mu t}{\sigma t^{1/2}}\right).$$

In Problem 4.12 this formula will be verified by independent means, using the fact that (2) satisfies

$$(3) \quad \frac{\partial}{\partial t} G(t, x, y) = \left(\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x}\right) G(t, x, y) \quad \text{main equation,}$$

$$(4) \quad G(t, 0, y) = 0 \quad \text{boundary condition,}$$

$$(5) \quad G(0, x, y) = 1_{(x > y)} \quad \text{initial condition.}$$

Let $u: R \rightarrow R$ be a continuous cost function satisfying (3.1) as before. Building on earlier calculations, we now compute

$$(6) \quad g(x) \equiv E_x\left[\int_0^T e^{-\lambda t} u(X_t) dt\right], \quad x \in R.$$

This represents the *expected discounted cost incurred up to the time of absorption*. The first thing to note is that

$$(7) \quad g(x) = f(x) - E_x\left[\int_T^\infty e^{-\lambda t} u(X_t) dt; T < \infty\right],$$

where $f(x)$ is the *infinite-horizon* expected discounted cost calculated in §3. On $\{T < \infty\}$, let $X_t^* \equiv X_{T+t}$ and note that

$$(8) \quad \int_T^\infty e^{-\lambda t} u(X_t) dt = e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt.$$

To compute the second term on the right side of (7), we use the strong Markov property (0.5) with the particular functional

$$(9) \quad F(X) \equiv \int_0^\infty e^{-\lambda t} u(X_t) dt.$$

Specifically, (8), (9), and (0.5) give us

$$\begin{aligned} (10) \quad E_x\left[\int_T^\infty e^{-\lambda t} u(X_t) dt; T < \infty\right] &= E_x\left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt; T < \infty\right] \\ &\equiv \int_{\{T < \infty\}} \left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt\right] dP_x \\ &= \int_{\{T < \infty\}} E_x\left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt \mid \mathcal{F}_T\right] dP_x \\ &= \int_{\{T < \infty\}} e^{-\lambda T} E_x\left[\int_0^\infty e^{-\lambda t} u(X_t^*) dt \mid \mathcal{F}_T\right] dP_x \\ &= \int_{\{T < \infty\}} e^{-\lambda T} E_x[F(X^*) | \mathcal{F}_T] dP_x \\ &= \int_{\{T < \infty\}} e^{-\lambda T} f(X_T) dP_x \\ &= f(0) \int_{\{T < \infty\}} e^{-\lambda T} dP_x = f(0) e^{-\alpha_+(\lambda)x}. \end{aligned}$$

The last equality in (10) uses Proposition (2.23). We summarize all of this as follows.

(11) Proposition. $g(x) = f(x) - f(0)e^{-\alpha_+(\lambda)x}$, $x \geq 0$.

From (2.32), (2.33), and (3.9) it is found that g satisfies

$$(12) \quad \lambda g - \Gamma g = u \text{ on } (0, \infty)$$

and

$$(13) \quad g(0) = 0.$$

Our solution of the inhomogeneous equation (12) with boundary condition (13) has a form familiar to students of differential equations. It is built from a *particular solution* f of the main equation (12) plus a function $\theta_*(x) = \exp\{-\alpha_*(\lambda)x\}$ satisfying the *homogeneous* equation $\lambda\theta_* - \Gamma\theta_* = 0$ with boundary condition $\theta_*(0) = 1$. (There is also the question of boundary conditions at infinity, but we need not go into this at present.) In the next section, a probabilistic solution will be derived for the analogous problem on a finite interval.

§5. TWO ABSORBING BARRIERS

Fixing $b > 0$ as in §2, let us again set $T \equiv T(0) \wedge T(b)$ and restrict our attention to starting states $x \in [0, b]$. Under P_x the process $\{X(T \wedge t), t \geq 0\}$ is a (μ, σ) Brownian motion with starting state x and two absorbing barriers. The time-dependent distribution $P_x\{X(T \wedge t) \leq y\}$ is known only as an infinite sum, but again one can derive a fairly simple formula for the expected discounted cost incurred before absorption. Fix $\lambda > 0$, let $u: R \rightarrow R$ be a continuous cost function satisfying (3.1) as before, and define

$$(1) \quad h(x) \equiv E_x \left[\int_0^T e^{-\lambda t} u(X_t) dt \right] \\ = E_x \left[\int_0^\infty e^{-\lambda t} u(X_t) dt \right] - E_x \left[\int_T^\infty e^{-\lambda t} u(X_t) dt \right].$$

The first term on the right side of (1) is the quantity $f(x)$ calculated in §3, whereas the second term can be expressed as

$$(2) \quad E_x \left[\int_T^\infty e^{-\lambda t} u(X_t) dt; X_T = 0 \right] + E_x \left[\int_T^\infty e^{-\lambda t} u(X_t) dt; X_T = b \right].$$

Proceeding exactly as in (4.9), we define $X_t^* = X_{T+t}$ and use the strong Markov property (0.5) to conclude that

$$E_x \left[\int_T^\infty e^{-\lambda t} u(X_t) dt; X_T = 0 \right] \\ = E_x \left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt; X_T = 0 \right]$$

$$\begin{aligned} &= \int_{\{X(T)=0\}} \left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt \right] dP_x \\ &= \int_{\{X(T)=0\}} E_x \left[e^{-\lambda T} \int_0^\infty e^{-\lambda t} u(X_t^*) dt \mid \mathcal{F}_T \right] dP_x \\ &= \int_{\{X(T)=0\}} e^{-\lambda T} f(0) dP_x \equiv f(0)\psi_*(x). \end{aligned}$$

In the same way, the second term in (2) reduces to $f(b)\psi^*(x)$, and we arrive at the following.

(3) Proposition. The expected discounted cost $h(x)$ defined in (1) is given by $h(x) = f(x) - f(0)\psi_*(x) - f(b)\psi^*(x)$, $0 \leq x \leq b$, where $\psi_*(\cdot)$ and $\psi^*(\cdot)$ are given by (2.18).

From (2.34), (2.35), and (3.9) it is found that h satisfies

$$(4) \quad \lambda h - \Gamma h = u \text{ on } (0, b)$$

and

$$(5) \quad h(0) = h(b) = 0.$$

§6. MORE ON REGULATED BROWNIAN MOTION

Restricting attention again to positive starting states x , let us form processes L and Z by applying to X the one-sided regulator of §2.2. That is, let

$$L_t = \sup_{0 \leq s \leq t} X_s^- \quad \text{and} \quad Z_t = X_t + L_t \quad \text{for } t \geq 0,$$

implying that Z is a continuous process with $Z_0 = x$ under P_x . In §1.9 we calculated $P_0\{Z_t \leq y\}$ using a time reversal argument. Using the joint distribution computed in §1.6, one can generalize this to show that

$$(1) \quad Q(t, x, y) \equiv P_x\{Z_t > y\} \\ = \Phi\left(\frac{-y + x + \mu t}{\sigma t^{1/2}}\right) + e^{2\mu y/\sigma^2} \Phi\left(\frac{-y - x - \mu t}{\sigma t^{1/2}}\right)$$

for $x, y, t \geq 0$. As with the expression for $G(t, x, y)$ derived in §4, we shall later verify (1) by independent means (see Problem 5.12). In preparation, readers are asked in Problem 11 to verify that

- (2) $\frac{\partial}{\partial t} Q(t, x, y) = \left(\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) Q(t, x, y)$ main equation,
- (3) $\frac{\partial}{\partial x} Q(t, 0, y) = 0$ boundary condition,
- (4) $Q(0, x, y) = 1_{(x > y)}$ initial condition.

The mysterious thing here is the boundary condition (3) whose explanation must await the development of stochastic calculus.

PROBLEMS AND COMPLEMENTS

1. Verify that the transition density $p(t, x, y)$ given by (1.1) satisfies the backward equation (1.2) and forward equation (1.4). Use the fact that $\phi'(z) = -z\phi(z)$.
2. Let $l(t, y)$ be the local time at level y of the (μ, σ) Brownian motion X as in §1.3, and let $u: R \rightarrow R$ be bounded and continuous. Take E_x of both sides in (1.3.8) and use Fubini's theorem to conclude that

$$\int_0^t E_x[u(X_s)] ds = \int_R u(y) E_x[l(t, y)] dy.$$

But $E_x[u(X_s)] = \int_R u(y) p(s, x, y) dy$ by the definition of the transition density p . Conclude that

$$p(t, x, y) = \frac{\partial}{\partial t} E_x[l(t, y)].$$

3. Verify that the functions θ_* , θ^* , ψ^* , and ψ_* given by (2.19) to (2.22) satisfy (2.32) to (2.35). In Problem 4.1 these differential equations and boundary conditions will be used to verify the calculations done in §2.
4. Working directly from (3.2), show that $\lambda^3 f(x) = \lambda^2 x^2 + 2x\mu\lambda + 2\mu^2 + \sigma^2\lambda$ if $u(y) = y^2$. Verify that the proposed solution for f satisfies $\lambda f - \Gamma f = u$ on R .

5. Let $u: R \rightarrow R$ be a continuous function satisfying (3.1), fix $\lambda > 0$, and define the expected discounted cost function $f(x)$ via (3.2). Working directly from (3.2), use Fubini's theorem to show that $f(x) = \int_R u(y) \pi(x, y) dy$, where

$$\pi(x, y) \equiv \int_0^\infty e^{-\lambda t} p(t, x, y) dt.$$

Observing that $p(t, y, y) = \phi(-\mu t^{1/2}/\sigma)/\sigma t^{1/2}$, show that

$$\pi(y, y) = (\mu^2 + 2\lambda\sigma^2)^{-1/2} \quad \text{for all } y \in R.$$

6. (**Continuation**) Let $l(t, y)$ be the local time of X as in Problem 2. Starting with the identity proved in Problem 2, show that

$$\pi(x, y) = E_x \left[\int_0^\infty e^{-\lambda t} l(dt, y) \right]$$

with the integral on the right defined path by path in the Riemann–Stieltjes sense.

7. (**Continuation**) Let $T \equiv T(y)$, let X^* be defined in terms of T on $\{T < \infty\}$ by (0.3), and let $l^*(t, y)$ be the local time at level y for the process X^* . Recall from §1.3 that $l(t, y) = 0$ for $0 \leq t \leq T$. Show that

$$\int_0^\infty e^{-\lambda t} l(dt, y) = e^{-\lambda T} \int_0^\infty e^{-\lambda t} l^*(dt, y) \text{ on } \{T < \infty\}.$$

8. (**Continuation**) Use the strong Markov property (0.5) and the results of Problems 6 and 7 to show that

$$(*) \quad \pi(x, y) = E_x[e^{-\lambda T(y)}] \pi(y, y).$$

Let $\theta(x, y)$ be defined by (3.6). From Proposition (2.23) we see that $\theta(x, y) = E_x[\exp\{-\lambda T(y)\}]$. Combining this with (*) and the result of Problem 5, we obtain the general formula (3.4) for the expected discounted cost $f(x)$.

9. Verify that π satisfies (3.7) and (3.8). Using this, verify that f satisfies the differential equation (3.9).
10. Verify that our solution (4.2) for $G(t, x, y)$ satisfies the partial differential equation (4.3) with boundary condition (4.4) and initial condition

(4.5). It is helpful to note that each term on the right side of (4.2) satisfies the main equation (4.3) separately.

11. Verify that our solution (6.1) for $Q(t, x, y)$ satisfies the partial differential equation (6.2) with boundary condition (6.3) and initial condition (6.4). Again it is helpful to note that each term satisfies the main equation separately.
12. Extend the proof of (2.1) to show that $E_x(T) = x(b - x)/\sigma^2$ if $\mu = 0$. For nonzero drift, show that

$$\mu E_x(T) = x - b \left[\frac{1 - \xi(x)}{1 - \xi(b)} \right],$$

using Proposition (2.27) and the fact that $X_t - \mu t$ is a martingale on $(\Omega, \mathbb{F}, P_x)$.

13. Fix $b > 0$, assume $0 \leq x \leq b$, and let $T = T(0) \wedge T(b)$ as in §2. Consider a process Z that coincides with X over $[0, T)$ but then jumps instantaneously to q or Q ($0 < q < Q < b$) depending on whether $X_T = 0$ or $X_T = b$. Thereafter Z repeats this behavior in a regenerative fashion as shown in Figure 3. Our purpose is to compute the expected discounted cost function

$$k(x) \equiv E_x \left[\int_0^\infty e^{-\lambda t} u(Z_t) dt \right], \quad 0 \leq x \leq b,$$

where $u: R \rightarrow R$ is bounded and measurable. Argue informally that

$$k(x) = h(x) + \psi_*(x)k(q) + \psi^*(x)k(Q),$$

where $h(\cdot)$, $\psi_*(\cdot)$, and $\psi^*(\cdot)$ are as computed earlier in §2 and §5.

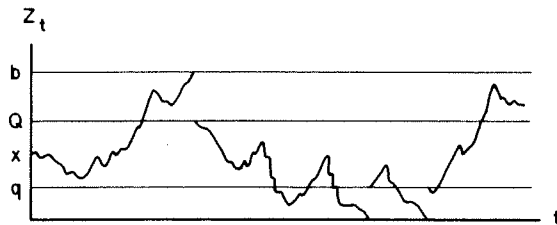


Figure 3. Brownian motion with jump boundaries.

Then show that $k(q)$ and $k(Q)$ are uniquely determined by the vector-matrix relation

$$\begin{bmatrix} k(q) \\ k(Q) \end{bmatrix} = \begin{bmatrix} h(q) \\ h(Q) \end{bmatrix} + \begin{bmatrix} \psi_*(q) & \psi^*(q) \\ \psi_*(Q) & \psi^*(Q) \end{bmatrix} \begin{bmatrix} k(q) \\ k(Q) \end{bmatrix}.$$

To make this rigorous, the strong Markov property (0.5) is used after spelling out more precisely how Z is constructed from X .

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Stochastic Calculus

It is the purpose of this chapter to state, in a form suitable for later applications, several variations of the Ito differentiation formula for Brownian motion. Because we seek to record only such information as is required for intelligent application of the Ito calculus, only a few propositions will be proved completely. Nonetheless a substantial development is required to just *state* the results of interest in precise mathematical terms. In particular, we must define what is meant by integration with respect to Brownian motion.

§0. INTRODUCTION

Departing from previous usage, let us denote by W a standard Brownian motion (or Wiener process) on some filtered probability space $(\Omega, \mathcal{F}, P)$. Readers should review the meaning given to this phrase in §1.2, recalling in particular that

- (1) $W(t + u) - W(t)$ is independent of $\mathcal{F}_t$ for all $t, u \geq 0$.

In addition to the standing assumptions enunciated in §A.1, it is assumed throughout this chapter that

- (2) the probability space $(\Omega, \mathcal{F}, P)$ is complete and
 (3) $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$.

Condition (2) means that if $A \in \mathcal{F}$, $P(A) = 0$, and $B \subseteq A$, then $B \in \mathcal{F}$ as well. (Of course, $P(B) = 0$ for all such B , for otherwise P would not be additive and thus would not be a probability measure.) As readers will see in §9,

these extra conditions are quite harmless; beginning with any filtered probability space, one can always augment the filtration in such a way that (2) and (3) are satisfied but the probabilistic model is unchanged. Our objective in §1 to 3 is to define a continuous stochastic process

$$(4) \quad I_t(X) \equiv \int_0^t X \, dW, \quad t \geq 0,$$

for a certain class of processes X . To be specific, let H denote the set of all integrands X such that

- (5) X is an adapted process on $(\Omega, \mathcal{F}, P)$ and
 (6) $P\left\{\int_0^t X^2(s) \, ds < \infty\right\} = 1$ for all $t \geq 0$.

We seek to give (4) a precise meaning for all $X \in H$. The random variables $I_t(X)$ will be called *stochastic integrals* (of X with respect to W), and the entire process $\{I_t(X), t \geq 0\}$ will be denoted by the indefinite integral $\int X \, dW$.

Remember that we only use the term *stochastic process* when referring to functions $X(\omega, t)$ that are jointly measurable in ω and t (see §A.2) and thus jointly measurability is implicit in (5). Conditions (1) and (5) together imply that $\{X(s), 0 \leq s \leq t\}$ and $\{W(t + u) - W(t), u \geq 0\}$ are independent for each t . This restriction is essential for the theory developed here, and the interested reader may consult page 31 of McKean (1969) to see why (6) is indispensable as well.

Suppose that X is a VF process, meaning that almost every sample path is a VF function in the sense of §B.2. Then the integrability theorem (B.3.3) shows that $I_t(X)$ can be defined for each fixed ω in the Riemann–Stieltjes sense. Unfortunately, our primary interest is in the case where X has unbounded variation like W ; hence the stochastic integral cannot be defined in any conventional sense. What will be shown in §1 to 3 is that $I_t(X)$ can be defined for integrands $X \in H$ by a limiting procedure that is probabilistically natural and intuitive.

There are now many mathematical books that develop the stochastic calculus for Brownian motion, often in the context of a more general theory. We shall adopt McKean (1969) as our standard reference, motivated by several considerations. First, McKean's book is mathematically correct (see comments in §3) and explicit. Second, it is widely used as a reference by pure mathematicians and applied researchers alike. Third, the approach is clean and efficient, in the sense that only two basic propositions need be stated to

complete the definition of $\int X dW$ for general $X \in H$. Finally, McKean develops only the stochastic integral with respect to Brownian motion, which has distinct advantages. Authors who allow a more general integrator must impose further restrictions on the integrand X , and those restrictions involve subtleties that are simply irrelevant for our purposes.

On the negative side, McKean's treatment of stochastic integrals is mathematically difficult and very terse. (There has been speculation that McKean's book was originally transcribed as a telegram.) To focus attention on the basic concepts, we begin by presenting Ito's original definition of $I_t(X)$ for individual time points t and integrands X that satisfy a stronger condition than (6). This definition is actually quite simple, and most of the steps will be proved. Section 2 is devoted to analysis of a revealing example, and then McKean's version of the general theory is laid out in §3. The basic Ito formula and various generalizations are presented in §4 to 8, and §9 gives some first applications.

§1. FIRST DEFINITION OF THE ITO INTEGRAL

Hereafter let H^2 be the set of all adapted processes X on $(\Omega, \mathbb{F}, P)$ satisfying

$$(1) \quad E \left[\int_0^t X^2(s) ds \right] < \infty \quad \text{for all } t \geq 0.$$

Condition (1) is stronger than (0.6) and so H^2 is a proper subset of H . A process X will be called *simple* if there exist times $\{t_k\}$ such that

$$(2) \quad 0 = t_0 < t_1 < \dots < t_k \rightarrow \infty$$

and

$$(3) \quad X(t, \omega) = X(t_{k-1}, \omega) \quad \text{for all } t \in [t_{k-1}, t_k) \text{ and } k = 1, 2, \dots$$

Note that the times $\{t_k\}$ do not depend on ω . Let S be the set of all simple adapted processes, and let S^2 be the set of all simple $X \in H^2$.

Let L^2 denote as usual the set of all random variables ξ on $(\Omega, \mathbb{F}, P)$ such that

$$(4) \quad \|\xi\| \equiv [E(\xi^2)]^{1/2} < \infty.$$

When we say that $\xi_n \rightarrow \xi$ in L^2 , this means that $\xi_1, \xi_2, \dots, \xi$ are all elements of L^2 and $\|\xi_n - \xi\| \rightarrow 0$. A sequence $\{\xi_n\}$ in L^2 is said to be *fundamental* (or

a Cauchy sequence) if $\|\xi_m - \xi_n\|$ can be made arbitrarily small by taking m and n sufficiently large. The following is a well-known result from analysis.

(5) Proposition. L^2 is complete. That is, every fundamental sequence has a limit in L^2 .

Fixing $t > 0$ until further notice, we now define $I_t(X)$ for $X \in H^2$. To begin, let

$$(6) \quad \|X\| \equiv E \left[\int_0^t X^2(s) ds \right]^{1/2} \quad \text{for } X \in H^2.$$

Although the same symbol $\|\cdot\|$ is used to denote a norm on L^2 and a norm on H^2 , attentive readers will find that this causes no serious confusion. Convergence of sequences in H^2 is defined just as for sequences in L^2 . The following proposition is important but rather technical, and hence we refer to pages 92–95 of Liptser–Shiryayev (1977) for its proof.

(7) Proposition. S^2 is dense in H^2 . That is, for each $X \in H^2$ there exist simple processes $\{X_n\}$ such that

$$(8) \quad X_n \rightarrow X \text{ in } H^2 \text{ as } n \rightarrow \infty.$$

To simplify notation, set $I(X) \equiv I_t(X)$ until t is freed. If X is simple, then one can define $I(X)$ in the Riemann–Stieltjes sense (see §B.3) for almost every ω . To be specific, let us suppose (without loss of generality) that (2) and (3) hold with $t = t_n$. Then the Riemann–Stieltjes theory defines

$$(9) \quad I(X) = \sum_{k=0}^{n-1} X(t_k) [W(t_{k+1}) - W(t_k)].$$

Remember that $I(X)$ is a random variable, although we suppress its dependence on ω in the usual way.

(10) Proposition. If $X \in S^2$, then $E[I(X)] = 0$ and $\|I(X)\| = \|X\|$.

Remark. The second part of the conclusion says that $I(X) \in L^2$ and that the L^2 norm of $I(X)$ equals the H^2 norm of X .

Proof. Again suppose (2) and (3) hold with $t = t_n$ and write $\mathcal{F}_k$ in place of $\mathcal{F}(t_k)$ to simplify notation. For the first part, we use (9), (0.1), and the adaptedness of X to write

$$\begin{aligned}
E(I(X)) &= \sum_{k=0}^{n-1} E\{X(t_k)[W(t_{k+1}) - W(t_k)]\} \\
&= \sum_{k=0}^{n-1} E\{E\{X(t_k)[W(t_{k+1}) - W(t_k)] | \mathcal{F}_k\}\} \\
&= \sum_{k=0}^{n-1} E\{X(t_k)E[W(t_{k+1}) - W(t_k) | \mathcal{F}_k]\} = 0.
\end{aligned}$$

For the second part, note that

$$\begin{aligned}
I^2(X) &= \sum_{k=0}^{n-1} X^2(t_k)[W(t_{k+1}) - W(t_k)]^2 \\
&\quad + 2 \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} X(t_j)X(t_k)[W(t_{j+1}) - W(t_j)][W(t_{k+1}) - W(t_k)].
\end{aligned}$$

Now take the expectation of each side, first conditioning on $\mathcal{F}_k$ in the k th term of the first sum and in the (j,k) th term of the second sum. The first three factors inside the double sum are all measurable with respect to $\mathcal{F}_k$, whereas the conditional expectation of the last factor given $\mathcal{F}_k$ is zero as above. Thus the double sum has zero expectation and we come down to

$$\begin{aligned}
E[I^2(X)] &= E\left[\sum_{k=0}^{n-1} X^2(t_k)E\{[W(t_{k+1}) - W(t_k)]^2 | \mathcal{F}_k\}\right] \\
&= E\left[\sum_{k=0}^{n-1} X^2(t_k)(t_{k+1} - t_k)\right] = E\left[\int_0^t X^2(s) ds\right] = \|X\|^2.
\end{aligned}$$

Because $E[I^2(X)] = \|I(X)\|^2$, this completes the proof. $\square$

(11) Proposition. Suppose $X \in H^2$. There exists a random variable $I(X) \in L^2$, unique up to a null set, such that $I(X_n) \rightarrow I(X)$ in L^2 for each simple sequence $\{X_n\}$ satisfying (8). Furthermore, $E[I(X)] = 0$ and $\|I(X)\| = \|X\|$.

Remark. The phrase *unique up to a null set* means that any two random variables fitting this description are equal almost surely. Combining (7) and (11), the stochastic integral $I(X)$ is defined up to a null set for each $X \in H^2$.

Proof. Let $\{X_n\}$ be a sequence of simple processes for which (8) holds. Then $\{X_n\}$ is a fundamental sequence in H^2 . Proposition (10) gives us

$$\|I(X_m) - I(X_n)\| = \|I(X_m - X_n)\| = \|X_m - X_n\|,$$

and hence $\{I(X_n)\}$ is a fundamental sequence in L^2 . Thus by (5) there exists some random variable $I(X) \in L^2$ such that $I(X_n) \rightarrow I(X)$. If $\{X'_n\}$ is any other sequence in S^2 for which (8) holds, then $\|X_n - X'_n\| \leq \|X_n - X\| + \|X - X'_n\| \rightarrow 0$, and another application of (10) gives

$$\|I(X'_n) - I(X_n)\| = \|I(X'_n - X_n)\| = \|X'_n - X_n\| \rightarrow 0.$$

Thus $\|I(X'_n) - I(X)\| \leq \|I(X'_n) - I(X_n)\| + \|I(X_n) - I(X)\| \rightarrow 0$, meaning that $I(X'_n) \rightarrow I(X)$ as well. This establishes the uniqueness of the stochastic integral $I(X)$. If $\xi_n \rightarrow \xi$ in L^2 , it is well known (and an easy consequence of the dominated convergence theorem) that $E(\xi_n) \rightarrow E(\xi)$ and $\|\xi_n\| \rightarrow \|\xi\|$. The last statement of Proposition (11) follows from this and from (10) and thus the proof is complete. $\square$

This completes the definition of $I_t(X)$ for $X \in H^2$ and a fixed time $t \geq 0$. To be precise, Proposition (11) associates with each time t an *equivalence class* of L^2 random variables; any two members of this class are equal almost surely. It has *not* been shown that one can select a member of this class for each time t in such a way that $\int X dW$ is a continuous process.

§2. AN EXAMPLE AND SOME COMMENTARY

We now consider the integrand $X = W$, seeking to compute $I_t(W) \equiv \int_0^t W dW$ explicitly. Using Fubini's theorem (see §A.5), one notes first that

$$(1) \quad E\left[\int_0^t W^2(s) ds\right] = \int_0^t E[W^2(s)] ds = \int_0^t s ds = \frac{t^2}{2} < \infty,$$

and thus $W \in H^2$. Now fix $t > 0$ and consider the simple functions $\{X_n\}$ defined by

$$(2) \quad X_n(s) \equiv W\left(\frac{kt}{2^n}\right) \text{ for } s \in \left[\frac{kt}{2^n}, \frac{(k+1)t}{2^n}\right)$$

and $k = 0, 1, \dots$. Check that $X_1, X_2, \dots$ are adapted and are furthermore elements of H^2 . Moreover, defining the H^2 norm as in §1,

$$\begin{aligned}
 (3) \quad \|W - X_n\| &= E \left\{ \int_0^t [W(s) - X_n(s)]^2 ds \right\} \\
 &= \int_0^t E\{[W(s) - X_n(s)]^2\} ds \\
 &= \sum_{k=0}^{2^n-1} \int_0^{t/2^n} E \left\{ \left[W\left(\frac{kt}{2^n} + s\right) - W\left(\frac{kt}{2^n}\right) \right]^2 \right\} ds \\
 &= \sum_{k=0}^{2^n-1} \int_0^{t/2^n} s ds = 2^n(1/2) \left(\frac{t}{2^n}\right)^2 = \frac{t^2}{2^{n+1}}.
 \end{aligned}$$

Thus $\|W - X_n\| \rightarrow 0$, implying by (1.11) that $I_t(X_n) \rightarrow I_t(W)$. Fixing n for the moment, write t_k in place of $kt/2^n$. Specializing (1.9) to the simple processes X_n gives

$$\begin{aligned}
 (4) \quad I_t(X_n) &= \sum_{k=0}^{2^n-1} W(t_k)[W(t_{k+1}) - W(t_k)] \\
 &= \frac{1}{2} \sum_{k=0}^{2^n-1} \{[W^2(t_{k+1}) - W^2(t_k)] - [W^2(t_{k+1}) + W^2(t_k)] \\
 &\quad + 2W(t_k)W(t_{k+1})\} \\
 &= \frac{1}{2} \sum_{k=0}^{2^n-1} [W^2(t_{k+1}) - W^2(t_k)] - \frac{1}{2} \sum_{k=0}^{2^n-1} [W(t_{k+1}) - W(t_k)]^2 \\
 &= \frac{1}{2} W^2(t) - \frac{1}{2} \sum_{k=0}^{2^n-1} [W(t_{k+1}) - W(t_k)]^2.
 \end{aligned}$$

In §1.3 (dealing with quadratic variation) it was shown that the summation in the last line (a random variable) converges in the L^2 sense to t . Combining all this gives

$$(5) \quad I_t(X_n) \rightarrow \frac{1}{2} W^2(t) - \frac{1}{2} t \quad \text{in the } L^2 \text{ sense,}$$

and consequently

$$(6) \quad \int_0^t W dW = \frac{1}{2} W^2(t) - \frac{1}{2} t.$$

If W were a continuous VF process with $W(0) = 0$, formula (B.4.4) would give us $W^2(t) = 2 \int_0^t W dW$, with the integral defined in the Riemann–Stieltjes sense. Thus the surprising thing about (6) is the term $-t/2$ on the right. This peculiarity, of course, traces to the infinite variation of Brownian paths, and more particularly to their positive quadratic variation. Consider now simple processes $\{X'_n\}$ defined by

$$(7) \quad X'_n(s) = W\left(\frac{(k+1)t}{2^n}\right) \quad \text{for } s \in \left[\frac{kt}{2^n}, \frac{(k+1)t}{2^n}\right)$$

and $k = 0, 1, \dots$. In contrast with (2), this scheme approximates W over each interval $[kt/2^n, (k+1)t/2^n]$ by its value at the *right* endpoint. If X'_n is substituted for X_n in (4), one ultimately finds that $I_t(X'_n) \rightarrow \frac{1}{2} W^2(t) + t/2$ as $n \rightarrow \infty$. If W were Riemann–Stieltjes integrable with respect to itself, the substitution of $\{X'_n\}$ for $\{X_n\}$ would make no difference as $n \rightarrow \infty$, but we find that this substitution makes a great deal of difference when W is a Brownian motion. The key point here is that the simple processes $\{X'_n\}$ are *not* adapted and hence are not elements of S^2 ; they cannot be used in approximating $I_t(W)$. The approximating simple functions $\{X_n\}$ are elements of S^2 and hence (6) gives the correct value of $I_t(W)$ in Ito's stochastic calculus.

§3. FINAL DEFINITION OF THE INTEGRAL

We now state the formal definition of $\int X dW$ to be used hereafter. In the following key proposition, interpret (2) as meaning that, for almost all ω , the indicated inequality holds for all sufficiently large n .

(1) Proposition. Fix $X \in H$. For any $t > 0$ there exist simple adapted processes $\{X_n\}$ such that

$$(2) \quad P \left\{ \int_0^t [X_n(s) - X(s)]^2 ds \leq (\tfrac{1}{2})^n \text{ as } n \uparrow \infty \right\} = 1.$$

On page 23, McKean (1969) constructs a sequence of simple processes satisfying (2), but he does not prove that these processes are adapted. The gap is filled by Lemma 3.8 of Chung–Williams (1983) whose proof depends critically on assumptions (0.2) and (0.3). (This is the only point in the development of stochastic calculus in which those added assumptions come into play.) Given a sequence of simple adapted processes $\{X_n\}$ satisfying (2),

let us define $Z_n \equiv \int X_n dW$ in the Riemann–Stieltjes sense as in §1. Then Z_n is continuous and adapted for each n . The next proposition is proved on page 24 of McKean (1969).

(3) Proposition. There exists a process Z , unique up to a null set, with the following property. If $t > 0$, $\{X_n\}$ is a sequence of simple integrands satisfying (2) and $Z_n \equiv \int X_n dW$, then

$$(4) \quad P \left\{ \sup_{0 \leq s \leq t} |Z_n(s) - Z(s)| \rightarrow 0 \text{ as } n \rightarrow \infty \right\} = 1.$$

Remark. Because Z_n converges *uniformly* to Z on $[0, t]$ for almost all ω , it is immediate that Z is adapted and continuous with $Z(0) = 0$.

Definition. When we refer to $\int X dW$ hereafter, this is understood to mean (for almost all ω) the continuous process Z in (3).

This definition of the stochastic integral, like that presented in §1, involves an approximating sequence of simple adapted integrands. What McKean shows is that one can choose simple approximations $\{X_n\}$ converging to X so fast, in the sense of (2), that the continuous processes $\int X_n dW$ converge *almost surely* and *uniformly* over a given finite interval. Thus $I_t(X)$ is defined simultaneously for all t in the interval, and a continuous version of $\int X dW$ is automatically obtained. Furthermore, the approach is valid for all $X \in H$, not just H^2 .

For fixed t and $X \in H^2$, our final definition of $I_t(X)$ agrees with the one given earlier in §1, although that fact is not obvious. Directly from the definition of the integral, we get $\int (aX_1 + bX_2) dW = a \int X_1 dW + b \int X_2 dW$; this linearity will be used often without further comment. The following proposition, which generalizes the last part of (1.11), gives the only other property of the integral we shall need; it is implied by properties (4) and (5) on pages 24–25 of McKean (1969).

(5) Proposition. Suppose $X \in H$, $Z = \int X dW$, and T is a stopping time. If

$$(6) \quad E \left[\int_0^T X^2(t) dt \right] < \infty,$$

then $E[Z(T)] = 0$ and $E[Z^2(T)] = E[\int_0^T X^2(t) dt]$.

(7) Corollary. If $\{X(t), 0 \leq t \leq T\}$ is bounded and $E(T) < \infty$, then $E[Z(T)] = 0$.

Both (5) and (7) will be referred to later as *zero expectation properties* of the stochastic integral. A more fundamental property is that $\int X dW$ is a martingale for $X \in H^2$ and is what is called a *local martingale* for all $X \in H$. See Chung–Williams (1983) for elaboration.

§4. SIMPLEST VERSION OF THE ITO FORMULA

We continue with the setup where W is a standard Brownian motion on the filtered probability space $(\Omega, \mathbb{F}, P)$. The term *Ito process* will be used here to mean a process Z representable in the form

$$(1) \quad Z(t) = Z(0) + \int_0^t X dW + A(t), \quad t \geq 0,$$

where $Z(0) \in \mathcal{F}_0$, $X \in H$, and A is a continuous, adapted VF process with $A(0) = 0$. Thus Z is itself continuous and adapted. (Actually, the term *Ito process* will be used later in a slightly broader sense, but this will cause no confusion.) Our definition is a bit more generous than usual; most writers impose the further requirement that A be absolutely continuous, meaning that

$$(2) \quad A(\omega, t) = \int_0^t Y(\omega, s) ds, \quad t \geq 0,$$

where Y is jointly measurable in t and ω , is adapted, and satisfies

$$(3) \quad \int_0^t |Y(s)| ds < \infty \quad \text{almost surely,} \quad t \geq 0.$$

All important results with the usual definition carry over to our more general setting, and the resulting gain is important for our purposes. Specifically, regulated Brownian motion is an Ito process according to our definition but not according to the standard one (see §1.9).

Hereafter, the second term on the right side of (1) will be called the *Brownian component* of Z , and A will be called the *VF component*. When we say that a process Z has an *Ito differential* $X dW + dA$, or simply write

$dZ = X dW + dA$, this is understood to be shorthand for the precise statement (1), and it is similarly understood that X and A meet the restrictions stated immediately after (1). Also, when we say that Z is an Ito process with differential $dZ = X dW + Y dt$, this is understood as shorthand for (1) and (2) together, with X and Y meeting all the necessary restrictions. Incidentally, when $dZ = X dW + Y dt$, the VF process $A = \int Y(s) ds$ is usually called the *drift component* of Z , but that terminology will not be used here.

We now give an exact and explicit statement of the Ito differentiation rule in its simplest form, followed by several equivalent statements of the rule that are less explicit but more compact. A sketch of the proof will then be given. A complete proof is given on pages 34–35 of McKean (1969) for the case where $dA = Y dt$, and this argument extends immediately to our setting. For a process X and function $\phi: R \rightarrow R$, we shall hereafter denote by $\phi(X)$ the entire process $\{\phi(X_t), t \geq 0\}$.

(4) Proposition. Suppose $f: R \rightarrow R$ is twice continuously differentiable and Z is an Ito process with $dZ = X dW + dA$. Then

$$(5) \quad f(Z_t) = f(Z_0) + \int_0^t [f'(Z)X] dW + \int_0^t f'(Z) dA + \frac{1}{2} \int_0^t [f''(Z)X^2] ds, \quad t \geq 0,$$

where the first integral on the right is defined as in §3, the second is defined path by path in the Riemann–Stieltjes sense (see §B.3), and the third is defined path by path in the Riemann sense.

First Remark. It is customary to express (5) more compactly as

$$(6) \quad df(Z) = f'(Z) dZ + \frac{1}{2} f''(Z) X^2 dt,$$

with the following conventions understood. First, of course, is the fact that any equation involving Ito differentials is shorthand for a precise statement in terms of stochastic integrals. Second, dZ is shorthand for $X dW + dA$ and we therefore separate the dW and dA terms that result from this substitution. Thus (6) can be written more explicitly as

$$df(Z) = f'(Z)X dW + f'(Z) dA + \frac{1}{2} f''(Z)X^2 dt.$$

Second Remark. An even more highly symbolic expression of (5), and one that has real advantages in its multidimensional form, is

$$(7) \quad df(Z) = f'(Z) dZ + \frac{1}{2} f''(Z) (dZ)^2.$$

Here it is understood that one computes $(dZ)^2$ as

$$(8) \quad (dZ)^2 = (X dW + dA)^2 = X^2(dW)^2 + 2X dW dA + (dA)^2,$$

and then computes the various products according to the multiplication table below. That is, only the first term on the right side of (8) is nonzero, and its value is $X^2 dt$, so (7) reduces to (6).

	dW	dA
dW	dt	0
dA	0	0

Third Remark. We saw in §B.4 that the Riemann–Stieltjes calculus yields $df(Z) = f'(Z) dZ$ if Z is a continuous VF process. So the novel feature of (6) or (7) is the second term on the right side. It will be seen shortly that this traces to the positive quadratic variation of Brownian paths. Also, the following example connects the mysterious second term with our earlier surprising discovery that $2\int_0^t W dW = W^2(t) - t$ in the Ito calculus (see §2). If $Z = W$ and $f(x) = x^2$, then (6) gives $dW^2 = 2W dW + dt$. In precise integral form, this says that $W^2(t) = 2\int_0^t W dW + t$.

Sketch of Proof. The traditional method of proof is by brute force, using Taylor's theorem. For the special case $Z = W$, the argument goes as follows. Fix $t > 0$ and let $0 = t_0 < \dots < t_n = t$ be a partition of $[0, t]$. Then

$$(9) \quad f(W_t) - f(0) = \sum_{k=0}^{n-1} [f(W_{t_{k+1}}) - f(W_{t_k})].$$

According to Taylor's theorem (with the exact form of the remainder), each term on the right can be written as

$$(10) \quad f(W_{t_{k+1}}) - f(W_{t_k}) = f'(W_{t_k})[W_{t_{k+1}} - W_{t_k}] + \frac{1}{2} f''(\xi_k)[W_{t_{k+1}} - W_{t_k}]^2,$$

where ξ_k lies in the interval between W_{t_k} and $W_{t_{k+1}}$. Because W is continuous we can then write $\xi_k(\omega) = W(\tau_k(\omega))$, where $t_k \leq \tau_k(\omega) \leq t_{k+1}$. Thus (9) becomes

$$(11) \quad f(W_t) - f(0) = \sum_{k=0}^{n-1} f'(W(t_k)) [W(t_{k+1}) - W(t_k)] \\ + \frac{1}{2} \sum_{k=0}^{n-1} f''(W(\tau_k)) [W(t_{k+1}) - W(t_k)]^2.$$

Note that the first term on the right is the stochastic integral of a simple adapted process that closely approximates $f'(W)$ if the partition is fine. Also the quadratic variation theorem of §1.3 suggests that the second sum on the right side of (11) will be closely approximated by

$$\sum_{k=0}^{n-1} f''(W(\tau_k)) (t_{k+1} - t_k)$$

if the partition is fine. Thus the following statement is hardly surprising. There exists a sequence of successively finer partitions for which

$$(12) \quad \sum_{k=0}^{n-1} f'(W(t_k)) [W(t_{k+1}) - W(t_k)] \rightarrow \int_0^t f'(W) dW$$

$$(13) \quad \sum_{k=0}^{n-1} f''(W(\tau_k)) [W(t_{k+1}) - W(t_k)]^2 \rightarrow \int_0^t f''(W) ds,$$

both statements holding in the almost sure sense. Substituting (12) and (13) in (11) gives

$$f(W_t) - f(0) = \int_0^t f'(W) dW + \frac{1}{2} \int_0^t f''(W) ds, \quad t \geq 0,$$

which is the specialization of (5) to the case under discussion.

§5. THE MULTIDIMENSIONAL ITO FORMULA

Let us now generalize our setup, assuming that $W_1, \dots, W_n$ are n independent Wiener processes on the filtered probability space $(\Omega, \mathbb{F}, P)$. We consider here a vector process $Z = (Z_1, \dots, Z_m)$ whose components can be represented in the form

$$(1) \quad Z_i(t) = Z_i(0) + \sum_{j=1}^n \int_0^t X_{ij} dW_j + A_i(t), \quad t \geq 0,$$

where $Z_i(0)$ is measurable with respect to $\mathcal{F}_0$, $X_{ij} \in H$, and A_i is a continuous, adapted VF process. This is the general form of a *multidimensional Ito process*. Note that each of the stochastic integrals on the right side of (1) is well defined by the development in §3 so there is nothing new as yet. In the obvious way, we write

$$(2) \quad dZ_i = \sum_{j=1}^n X_{ij} dW_j + dA_i \quad (i = 1, \dots, m)$$

as shorthand for (1). Assuming that the precise meaning of differential statements like (2) is now clear, we shall state the multidimensional Ito formula only in the compact symbolic form analogous to (4.7). The following is proved on page 44 of McKean (1969) for the case where $dA_i = Y_i dt$, and again the proof goes through to our more general setting with only trivial modifications.

(3) Proposition. Suppose that $f: R^m \rightarrow R$ is twice continuously differentiable, meaning that all the first-order partials f_i and second-order partials f_{ij} exist and are continuous. If Z satisfies (2), then $f(Z)$ is an Ito process with differential

$$(4) \quad df(Z) = \sum_{i=1}^m f_i(Z) dZ_i + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m f_{ik}(Z) dZ_i dZ_k,$$

where the products $dZ_i dZ_k$ are computed using (2) and the multiplication rules $dW_j dW_k = \delta_{jk} dt$, $dW_j dA_i = 0$, $dA_i dA_k = 0$. (Here $\delta_{jk} = 1$ if $j = k$ and $= 0$ otherwise.)

Formula (4) is analogous to (2.7) in its level of symbolism. To assure that the multiplication rule is clearly understood, readers should verify that (4) becomes

$$(5) \quad df(Z) = \sum_{i=1}^m \sum_{j=1}^n [f_i(Z) X_{ij}] dW_j + \sum_{i=1}^m f_i(Z) dA_i \\ + \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m \sum_{j=1}^n [f_{ik}(Z) X_{ij} X_{kj}] dt$$

upon substitution of (2) and simplification. This version of the multidimensional formula is analogous to (2.6). Assuming that the exact meaning of such differential statements is clear from §4, we shall not write out (5) any more explicitly. In the future we shall consistently use the symbolism of (4) both because of its compactness and because this version of the formula is so much easier to remember than (5), at least for those familiar with the multidimensional Taylor formula.

Proposition (3) says that a smooth function f of an Ito process Z is itself an Ito process, with differential given by (4). What if we form a new process Z^* via $Z_t^* \equiv \phi(Z_t, t)$? If ϕ is twice continuously differentiable as a function of $m + 1$ variables, then this situation is covered by (3). Furthermore, all the conclusions of (3) remain valid with slightly weaker assumptions on ϕ ; the *second-order* partials involving t need not be continuous or even exist. McKean (1969) actually presents the multidimensional Ito formula under this weaker hypothesis; readers may look there for further information.

§6. TANAKA'S FORMULA AND LOCAL TIME

If Z is a one-dimensional Ito process and f is twice continuously differentiable, we have seen in §4 that $f(Z)$ is also an Ito process, and its differential is given by (2.5). What if f is not so smooth? In this section we address that question for the special case where Z is a (μ, σ) Brownian motion on $(\Omega, \mathbb{F}, P)$. To introduce the basic ideas in a simple setting, consider first the case $f(x) = |x|$. More precisely, in an effort to approximate the absolute value by a smoother function, let $\epsilon > 0$ be arbitrary and define $f: \mathbb{R} \rightarrow \mathbb{R}$ via

$$(1) \quad f(0) = f'(0) = 0$$

and

$$(2) \quad f''(x) = \begin{cases} 1/\epsilon & \text{if } |x| \leq \epsilon \\ 0 & \text{otherwise} \end{cases}.$$

It follows that

$$(3) \quad f'(x) = \begin{cases} x/\epsilon & \text{if } |x| \leq \epsilon \\ \text{sgn}(x) & \text{otherwise,} \end{cases}$$

and

$$(4) \quad f(x) = \begin{cases} x^2/2\epsilon & \text{if } |x| \leq \epsilon \\ |x| - \frac{1}{2}\epsilon & \text{otherwise} \end{cases}.$$

Figure 1 shows f and its first two derivatives. If f had a *continuous* second derivative, then we could apply the basic Ito formula (4.4) to obtain

$$(5) \quad f(Z_t) = f(Z_0) + \int_0^t f'(Z) dZ + \frac{1}{2} \sigma^2 \int_0^t f''(Z) ds.$$

Furthermore, (5) remains valid for the function f defined by (4), as can be proved with an approximation argument. Substituting (2) into the last term of (5) and denoting by $l(t, x)$ the local time of Z at level x (see §1.3), we have

$$(6) \quad \frac{1}{2} \int_0^t f''(Z) ds = \frac{1}{2\epsilon} \int_0^t 1_{\{|Z(s)| \leq \epsilon\}} ds \rightarrow l(t, 0) \quad \text{as } \epsilon \downarrow 0.$$

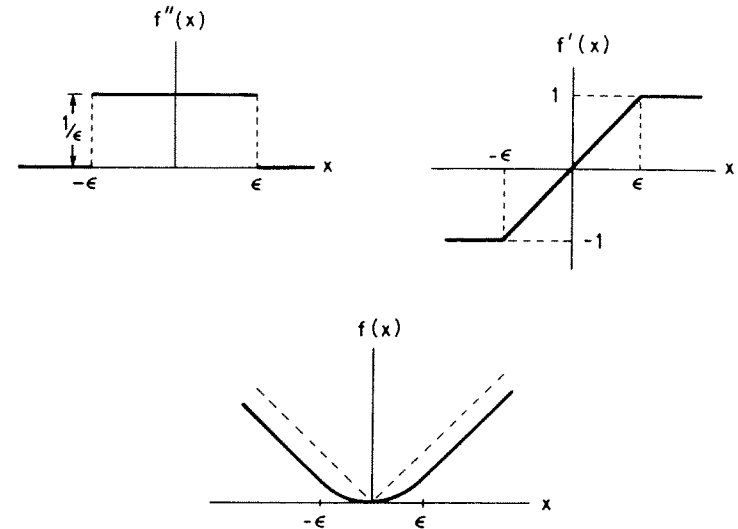


Figure 1. Approximating the absolute value function.

Furthermore, using the explicit formula (3), it is not difficult to show that the second term on the right side of (5) approaches $\int \text{sgn}(Z) dZ$ as $\epsilon \downarrow 0$. Of course, $f(Z_t) \rightarrow |Z_t|$ as $\epsilon \downarrow 0$ by (4). Combining this with (5) and (6), we have

$$(7) \quad |Z_t| = \int_0^t \text{sgn}(Z) dZ + \sigma^2 l(t, 0), \quad t \geq 0.$$

In the particular case where Z is a *standard* Brownian motion, (7) is called *Tanaka's formula*. It is stated (in a slightly different form) and proved on pages 68–69 of McKean (1969). To generalize further, let us introduce the following. (We write RCLL to indicate a right-continuous function for which left limits exist.)

(8) Assumption. Let $f: R \rightarrow R$ be absolutely continuous with RCLL density f' . It is assumed that f' has finite variation on every finite interval. Let ρ be the signed measure on $(R, \mathcal{B})$ defined by $\rho(a, b] = f'(b) - f'(a)$ for $-\infty < a < b < \infty$.

One may paraphrase (8) saying that the second derivative of f exists as a measure. A function satisfies this description if and only if it can be written as the difference of two convex functions. The following is proved in §9.2 of Chung–Williams (1983) for the case of standard Brownian motion, and the extension to general drift and variance parameters is trivial.

(9) Proposition. If Z is a (μ, σ) Brownian motion and f satisfies (8), then

$$(10) \quad f(Z_t) = f(Z_0) + \int_0^t f'(Z) dZ + \frac{1}{2} \sigma^2 \int_R l(t, x) \rho(dx).$$

If f is twice continuously differentiable, then $\rho(dx) = f''(x) dx$, and equation (1.3.8) specializes to give

$$(11) \quad \int_R l(t, x) \rho(dx) = \int_R l(t, x) f''(x) dx = \int_0^t f''(Z) ds,$$

and thus (10) reduces to the basic Ito formula (4.5) as it should. Because both equalities in (11) remain valid if f'' has discontinuities, we see that the basic Ito formula extends to this situation as well. On the other hand, Proposition (9) shows that fundamentally new effects enter if f' has discontinuities. In particular, the right side of (10) has a term involving $l(t, x)$ for each point x where f' jumps. In the case where $f(x) = |x|$, $\rho\{0\} = 2$ and

$\rho(dx) = 0$ away from the origin. Thus $\int l(t, x) \rho(dx) = 2l(t, 0)$, and (10) specializes to the Tanaka formula (7) as it should. Proposition (9) shows that $f(Z)$ is an Ito process in the sense that we use the term here (see §4) for any function f of the class (8).

§7. ANOTHER GENERALIZATION OF ITO'S FORMULA

Having studied the process $f(Z)$ when f is less smooth than required by the basic Ito formula, let us now see what happens if Z is less smooth than assumed in §4. In this section, Z is assumed to have the form

$$(1) \quad Z_t = \int_0^t X dW + V_t, \quad t \geq 0,$$

where $X \in H$ and

(2) V is an adapted, right-continuous VF process.

It follows that the left limit $V(t-)$ exists for all $t > 0$ almost surely and that V has just countably many points of discontinuity (or jumps) almost surely. Let us denote by $\Delta V(t) \equiv V(t) - V(t-)$ the jump of V at time t , and define a new process A via

$$(3) \quad A_t \equiv V_t - \sum_{0 < s \leq t} \Delta V_s, \quad t \geq 0,$$

where the sum is over the countable set of $s \in (0, t]$ at which $|\Delta V_s| > 0$. Incidentally, because V has VF sample paths, we know that

$$\sum_{0 < s \leq t} |\Delta V_s| < \infty$$

almost surely, so the sum in (3) makes sense. Obviously A is a continuous VF process; we call it the *continuous part* of V .

(4) Proposition: Suppose $f: R \rightarrow R$ is twice continuously differentiable and Z has the form (1) and (2). Then

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z) X dW + \int_0^t f'(Z) dA$$

$$+ \frac{1}{2} \int_0^t f''(Z) X^2 ds + \sum_{0 < s \leq t} \Delta f(Z)_s ,$$

where

$$\Delta f(Z)_s \equiv f(Z_s) - f(Z_{s-}) \quad \text{for } t > 0 .$$

If V (and hence Z) jumps only at isolated points $0 < T_1 < T_2 < \dots \rightarrow \infty$, then (4) is just a trivial extension of the ordinary Ito formula; one can prove it by applying (4.5) on each of the intervals $[T_{n-1}, T_n)$ and adding. This observation can be combined with an approximation argument to prove (4) in generality, or (4) can be viewed as a special case of the change of variable formula for semimartingales that appears on page 301 of Meyer (1976). The generalized Ito formula (4) will play a major role in Chapter 7 where we study a problem of optimal stochastic control.

§8. INTEGRATION BY PARTS (SPECIAL CASES)

Let Y and Z be two Ito processes with differentials $dY = X dW + dA$ and $dZ = X^* dW + dA^*$, respectively. Note that Y and Z are built from a common standard Brownian motion W . Also, remember that our definition of Ito process requires that A and A^* be continuous VF processes. Let us apply the multidimensional Ito formula (5.4) to analyze the product $U_t \equiv Y_t Z_t$. Defining $f(y, z) \equiv yz$, we have

$$\frac{\partial}{\partial y} f = z, \quad \frac{\partial}{\partial z} f = y, \quad \frac{\partial^2}{\partial y^2} f = \frac{\partial^2}{\partial z^2} f = 0, \quad \frac{\partial^2}{\partial y \partial z} f = 1 .$$

Of course, $U_t = f(Y_t, Z_t)$, so (5.4) gives

$$(1) \quad dU = Y dZ + Z dY + (dY)(dZ) ,$$

where

$$(2) \quad (dY)(dZ) = XX^*(dW)^2 = XX^* dt .$$

Substituting (2) into (1) and writing out the precise integral form, we have

$$(3) \quad Y_t Z_t = Y_0 Z_0 + \int_0^t Y dZ + \int_0^t Z dY + \int_0^t XX^* ds .$$

If either $X = 0$ or $X^* = 0$, meaning that either Y or Z is a VF process, then (3) reduces to the ordinary Riemann–Stieltjes integration by parts theorem (see §B.3). The next proposition strengthens that statement slightly.

(4) Proposition. Suppose that $Y = \int X dW + V$, where V is an adapted and right-continuous VF process as in §7. If Z is a continuous VF process, then

$$(5) \quad Y_t Z_t = Y_0 Z_0 + \int_0^t Y dZ + \int_0^t Z dY, \quad t \geq 0 .$$

On the right side of (5), $\int Y dZ$ is interpreted as $\int (f X dW) dZ + \int V dZ$. The integrand in the first term is continuous (an Ito process) and the integrand in the second term is a VF process and thus each integral can be defined path by path in the Riemann–Stieltjes sense by Proposition (B.3.3). Similarly, $\int Z dY$ is interpreted as $\int ZX dW + \int Z dV$; the first term is a stochastic integral and the second is defined path by path in the Riemann–Stieltjes sense. If V can jump just finitely often in any finite interval, then (4) can be proved by simply applying (3) to periods between jumps. We can use this plus an approximation argument to prove (4) in generality or can view it as a special case of the integration by parts formula for semimartingales, which appears on page 303 of Meyer (1976). By specializing (4) to the case $Z_t = \exp(-\lambda t)$, we get the following proposition, which will be used frequently in later discussion of expected discounted costs for Brownian motion.

(6) Proposition. Let Y be as in (4). Then for any real constant λ and $t \geq 0$ we have

$$e^{-\lambda t} Y_t = Y_0 + \int_0^t e^{-\lambda s} dY - \lambda \int_0^t e^{-\lambda s} Y ds .$$

§9. DIFFERENTIAL EQUATIONS FOR BROWNIAN MOTION

In Chapter 3 we used probabilistic methods to compute various interesting quantities associated with Brownian motion. After the fact, it was observed that the quantity in question, viewed as function of starting state and perhaps time, satisfied a differential equation with certain auxiliary conditions. In this section, it will be shown how Ito's formula can be used to derive such differential equations directly. Thus probabilistic questions can be

recast in purely analytic terms and attacked with purely analytic methods. Some problems are most easily solved by such an approach, or by a blend of probabilistic and analytic methods, as will be seen in the next chapter.

As in §A.3, let $(\Omega, \mathcal{F}) = (C, \mathcal{C})$, let X be the coordinate process on Ω , and let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the filtration generated by X . Recall from §A.3 that our ambient σ -field is $\mathcal{F} = \mathcal{F}_\infty$. Given parameters μ and $\sigma > 0$, let P_x be as described in §3.0. Then X is a (μ, σ) Brownian motion with starting state x on the filtered probability space $(\Omega, \mathbb{F}, P_x)$. As it happens, assumptions (0.2) and (0.3) are *not* satisfied with this setup, but discussion of that issue will be postponed until the end of the section. Defining

$$W_t \equiv \frac{1}{\sigma}(X_t - X_0 - \mu t), \quad t \geq 0,$$

we observe that W is a standard Brownian motion on $(\Omega, \mathbb{F}, P_x)$. Also, X is an Ito process with Brownian component σW and VF component μt . Let $f: R \rightarrow R$ be twice continuously differentiable, and again define the differential operator Γ via

$$\Gamma f \equiv \frac{1}{2} \sigma^2 f'' + \mu f'.$$

(1) Proposition. $f(X)$ is an Ito process with differential $df(X) = \sigma f'(X) dW + \Gamma f(X) dt$.

Proof. From the basic Ito formula (4.5) we have

$$\begin{aligned} df(X) &= f'(X) dX + \frac{1}{2} f''(X)(dX)^2 \\ &= f'(X)(\sigma dW + \mu dt) + \frac{1}{2} f''(X)\sigma^2 dt \\ &= \sigma f'(X) dW + \Gamma f(X) dt. \quad \square \end{aligned}$$

(2) Proposition. Fixing $\lambda > 0$, let $u: R \rightarrow R$ be defined by $u \equiv \lambda f - \Gamma f$. Let $a, b \in R$ be such that $a < x < b$, and define the stopping time $T \equiv T(a) \wedge T(b)$ as in §3.2. Then

$$(3) \quad f(x) = E_x \left[\int_0^T e^{-\lambda t} u(X_t) dt \right] + E_x[e^{-\lambda T} f(X_T)].$$

Remark. In Problems 1 to 3, the fundamental identity (3) will be used to verify certain calculations done earlier in Chapter 3.

Proof. From (1) we know that $df(X) = \sigma f'(X) dW + \Gamma f(X) dt$. Applying the specialized integration by parts formula (8.6) with $f(X)$ in place of Y ,

$$\begin{aligned} (4) \quad e^{-\lambda t} f(X_t) &= f(X_0) + \int_0^t e^{-\lambda s} df(X) - \lambda \int_0^t e^{-\lambda s} f(X) ds \\ &= f(X_0) + \sigma \int_0^t e^{-\lambda s} f'(X) dW + \int_0^t e^{-\lambda s} (\Gamma f - \lambda f)(X) ds. \end{aligned}$$

Defining

$$(5) \quad M_t \equiv \sigma \int_0^t e^{-\lambda s} f'(X) dW,$$

we can rewrite (4) as

$$(6) \quad e^{-\lambda t} f(X_t) = f(X_0) + M_t - \int_0^t e^{-\lambda s} u(X_s) ds.$$

Because (6) is a sample path relationship (an almost sure equality between two continuous processes), it remains valid when T is substituted for t . Taking E_x of both sides then gives

$$(7) \quad E_x[e^{-\lambda T} f(X_T)] = f(x) + E_x(M_T) - E_x \left[\int_0^T e^{-\lambda t} u(X_t) dt \right].$$

The continuous function f' is bounded over $[a, b]$, so the integrand in (5) is bounded over the time interval $[0, T]$. It was shown in §3.2 that $E_x(T) < \infty$, so the zero expectation property (3.7) gives $E_x(M_T) = 0$. When this is substituted in (7), the proof is complete. $\square$

As stated earlier, assumptions (0.2) and (0.3) are not satisfied under the setup used in this section. Because these assumptions were used in the limiting procedure that defines $\int X dW$ (see §3), there is no guarantee that the stochastic integrals appearing in Proposition 1 and the proof of Proposition 2 are well defined. Nonetheless Proposition 2 is true exactly as stated, and the path to this conclusion can be rigorized as follows. Fix a starting state x and let $\mathcal{F}^*$ consist of all $A \subseteq \Omega$ such that $A_1 \subseteq A \subseteq A_2$, where A_1 and A_2 are events in $\mathcal{F}$ with $P_x(A_1) = P_x(A_2)$. Then $\mathcal{F}^*$ is a σ -algebra, and we extend P_x to $\mathcal{F}^*$ in the obvious way, setting $P_x(A) = P_x(A_1) =$

$P_x(A_2)$. The probability space $(\Omega, \mathcal{F}^*, P_x)$ is complete as readers may verify. Now for each $t \geq 0$ let $\mathcal{F}_t^*$ be the smallest σ -algebra on Ω containing both $\mathcal{F}_t$ and all events $A \in \mathcal{F}^*$ such that $P_x(A) = 0$. This yields an augmented filtration $\mathbb{F}^* = \{\mathcal{F}_t^*, t \geq 0\}$, and the augmented space $(\Omega, \mathbb{F}^*, P_x)$ satisfies (0.2) and (0.3). To see that nothing has really been changed from a modeling standpoint, readers may verify the following: For every event $A \in \mathcal{F}_t^*$ there exists a $B \in \mathcal{F}_t$ such that $P_x(A \Delta B) = 0$, where $A \Delta B = A \cup B - A \cap B$. Loosely phrased, events in $\mathcal{F}_t^*$ differ from events in $\mathcal{F}_t$ only by null sets. From this it follows that $X(t+u) - X(t)$ is independent of $\mathcal{F}_t^*$ for each $t, u \geq 0$ (we have not added to $\mathcal{F}_t$ any information that foretells the future evolution of X) and hence that X is a (μ, σ) Brownian motion on the augmented space $(\Omega, \mathbb{F}^*, P_x)$. Because all the results of this chapter may now be invoked, Proposition 1 is rigorously established in the richer setting and one eventually arrives at Proposition 2.

Of course, Proposition 2 is simply a statement of equality between real numbers; the augmentation described above is needed only to ensure that all steps in the logical chain can be rigorously justified on the basis of previous results. There will be other places in future chapters where a similar augmentation is necessary to justify the use of results stated in this chapter, but no mention will be made of the matter. Those readers who realize the need for additional justification will know how to provide it, and those who forget will not get into trouble.

PROBLEMS AND COMPLEMENTS

1. In the setting of §9, suppose that f satisfies $\lambda f - \Gamma f = 0$ on $[a, b]$ with $f(a) = 1$ and $f(b) = 0$. Show that $f(x) = E_x\{\exp(-\lambda T); X_T = a\}$. When combined with Problem 3 of Chapter 3, this verifies the formula for $\psi_*(x)$ derived in §3.2, and the formula for $\psi^*(x)$ can be verified in the same way.
2. In the setting of §9, let $\psi_*(x) \equiv E_x\{\exp(-\lambda T); X_T = a\}$ and $\psi^*(x) \equiv E_x\{\exp(-\lambda T); X_T = b\}$. (This generalizes slightly the notation of §3.2, which was restricted to the case $a = 0$.) Show that

$$E_x[e^{-\lambda T} f(X_T)] = f(a)\psi_*(x) + f(b)\psi^*(x).$$

If $|f|$ is bounded by a polynomial, then the right side goes to zero as $a \rightarrow -\infty, b \rightarrow \infty$. Prove this statement, using the formulas for ψ_* and ψ^* developed in §3.2.

3. (**Continuation**) Let $u : R \rightarrow R$ be continuous with $|u|$ bounded by a polynomial. Suppose that f satisfies

$$(*) \quad \lambda f - \Gamma f = u$$

on $[a, b]$ and $f(a) = f(b) = 0$. Show that

$$f(x) = E_x \left[\int_0^T e^{-\lambda t} u(X_t) dt \right].$$

Next, dropping the requirement that $f(a) = f(b) = 0$, suppose that $(*)$ holds on all of R and that $|f|$ is bounded by a polynomial. Show that

$$f(x) = E_x \left[\int_0^\infty e^{-\lambda t} u(X_t) dt \right].$$

4. Altering slightly the setup in §9, suppose that f is continuous, that f is twice continuously differentiable except at an isolated point y ($a < y < b$), that $\Gamma f(x) = 0$ except at $x = y$, that $\sigma^2 \Delta f'(x) = -2$ (see §3.3 for the exact meaning of this condition), and that $f(a) = f(b) = 0$. Use the generalized Ito formula (6.10) to show that

$$f(x) = E_x[l(T, y)].$$

5. (**Continuation**) With $\lambda > 0$, now suppose that $\lambda f - \Gamma f = 0$ except at $x = y$. All other assumptions are as before. Show that

$$f(x) = E_x \left[\int_0^T e^{-\lambda t} l(dt, y) \right].$$

This requires that (6.10) be combined with the specialized integration by parts formula (8.6); the structure of the argument is the same as that used to prove Proposition (9.2).

6. Use Ito's formula to explicitly calculate $\int W^y dW$. Any expression not involving a stochastic integral is considered an answer.
7. Note that the key identity (9.3) remains valid when $\lambda = 0$. Suppose $\Gamma f = -1$ on $[0, b]$ with $f(0) = f(b) = 0$. Show that $f(x) = E_x(T)$. Show that the expression for $E_x(T)$ developed in Problem 3.12 does in fact satisfy this differential equation and these boundary conditions.
8. In the setting of §9, let $f_n(x) \equiv E_x[\int_0^T X_t^n dt]$. Use Ito's formula to develop a general formula for $f_n(x)$.
9. Let $f(t, x)$ be twice continuously differentiable on R^2 . Let μ and $\sigma > 0$ be constants and define

$$g(t, x) \equiv \frac{\partial}{\partial x} f(t, x)$$

and

$$h(t, x) \equiv \left(\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) f(t, x).$$

Adopting the setup of §9, let $T < \infty$ be a stopping time. Use the multidimensional Ito formula (5.4) to show that

$$f(T, X_T) = f(0, X_0) + \sigma \int_0^T g(t, X_t) dW + \int_0^T h(t, X_t) dt.$$

10. **(Continuation)** Fix $t > 0$, assume that $X_0 > 0$, and set $T \equiv T(0) \wedge t$. Suppose that $g(s, y)$ is bounded on $[0, t] \times [0, \infty)$ and that $h(s, y) = 0$ on $[0, t] \times [0, \infty)$. Use Corollary (3.7) to show that

$$f(0, x) = E_x[f(T, X_T)].$$

Finally, assume that $f(x, 0) = 0$ for $0 \leq s \leq t$ and conclude that

$$f(0, x) = E_x[f(t, X_t); T(0) > t].$$

11. **(Continuation)** Suppose that $G(t, x)$ is defined on $[0, \infty) \times [0, \infty)$ and is twice continuously differentiable up to the boundary. That is, all first- and second-order partials approach finite limits at all boundary points and those limits are continuous functions on the boundary; this condition assures that G can be extended to a function twice continuously differentiable on all of R^2 . Let $u: R \rightarrow R$ be bounded and continuous, and suppose that G satisfies

$$(a) \quad \frac{\partial}{\partial t} G(t, x) = \left(\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) G(t, x) \quad \text{for } t, x \geq 0,$$

$$(b) \quad G(t, 0) = 0 \quad \text{for } t \geq 0,$$

$$(c) \quad G(0, x) = u(x) \quad \text{for } x \geq 0,$$

$$(d) \quad \frac{\partial}{\partial x} G(t, x) \text{ is bounded on } [0, \infty) \times [0, \infty).$$

Now fix $t > 0$ and define $f(s, x) = G(t - s, x)$ for $0 \leq s \leq t$ and $x \geq 0$, observing that f can be extended to a function on R^2 satisfying all the conditions of Problem 10. Use the result of Problem 10 to conclude that

$$G(t, x) = E_x[u(X_t); T(0) > t].$$

12. **(Continuation)** Fix $y \geq 0$ and let $G(t, x, y)$ be defined by formula (3.4.2). Viewed as a function of t and x alone, this particular G does *not* satisfy the assumptions of Problem 11 because of discontinuities at $t = 0$. But for any $\epsilon > 0$ the function $G^*(t, x) \equiv G(t + \epsilon, x, y)$ does satisfy all the stated conditions (see Problem 3.10); apply the result of Problem 11 to conclude that

$$G^*(t, x) = E_x[G^*(0, X_t); T(0) > t],$$

or equivalently

$$G(t + \epsilon, x, y) = E_x[G(\epsilon, X_t, y); T(0) > t].$$

Recalling that $G(0, x, y) = 1_{(x > y)}$, let $\epsilon \downarrow 0$ and use the bounded convergence theorem to conclude that

$$G(t, x, y) = P_x\{X_t > y; T(0) > t\}.$$

This verifies the interpretation for G given in §3.4.

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Regulated Brownian Motion

In this chapter we study the stochastic processes (L, U, Z) that are obtained by applying the two-sided regulator to Brownian motion. The role of (L, U, Z) as a flow system model was discussed earlier in §2.6. Expected discounted costs will be calculated, and the steady-state distribution of Z will be determined, after certain fundamental properties have been established.

§1. STRONG MARKOV PROPERTY

Given parameters μ and $\sigma > 0$, let $(\Omega, \mathbb{F}, P_x)$ be the canonical space described in §3.0 and let X be the coordinate process on Ω . Thus X is a (μ, σ) Brownian motion with starting state x on $(\Omega, \mathbb{F}, P_x)$. Now let $b > 0$ be another fixed parameter and let (f, g, h) be the two-sided regulator defined in terms of b in §2.4. Restricting attention to starting states $x \in [0, b]$, we define processes $L \equiv f(X)$, $U \equiv g(X)$, and $Z \equiv h(X)$. The definition of the regulator says that

- (1) L and U are increasing and continuous with $L_0 = U_0 = 0$,
- (2) $Z_t \equiv X_t + L_t - U_t \in [0, b]$ for all $t \geq 0$, and
- (3) L and U increase only when $Z = 0$ and $Z = b$, respectively.

In §2.4 it was seen that the regulator (f, g, h) has a certain memoryless property. Combining this with the strong Markov property of Brownian motion gives the following important result. By way of setup, let T be an arbitrary stopping time and set

STRONG MARKOV PROPERTY

$$(4) \quad Z_t^* \equiv Z_{T+t}, \quad t \geq 0,$$

$$(5) \quad L_t^* \equiv L_{T+t} - L_T, \quad t \geq 0,$$

and

$$(6) \quad U_t^* \equiv U_{T+t} - U_T, \quad t \geq 0$$

on $\{T < \infty\}$; these processes will remain undefined on $\{T = \infty\}$. Also, let K be a measurable mapping $(C \times C \times C, \mathcal{C} \times \mathcal{C} \times \mathcal{C}) \rightarrow (R, \mathcal{B})$ such that $E_x(|K(L, U, Z)|) < \infty$ for all $x \in [0, b]$.

(7) Proposition. Let $k(x) \equiv E_x[K(L, U, Z)]$, $0 \leq x \leq b$. For each $x \in [0, b]$ we have

$$(8) \quad E_x[K(L^*, U^*, Z^*) | \mathcal{F}_T] = k(Z_T) \text{ on } \{T < \infty\}.$$

Remark. For lack of a better name, (8) will be referred to hereafter as the strong Markov property of (L, U, Z) . This terminology is not standard.

Proof. Fix $x \in [0, b]$. Random variables will be defined only on $\{T < \infty\}$, and identities between random variables will be understood as almost sure relations under P_x . For purposes of this proof, let us define

$$(9) \quad X_t^* \equiv Z_T + (X_{T+t} - X_T), \quad t \geq 0.$$

From the memoryless property (2.4.13) it follows directly that

$$(10) \quad L^* = f(X^*), \quad U^* = g(X^*), \quad \text{and} \quad Z^* = h(X^*).$$

That is, the triple (L^*, U^*, Z^*) is obtained by applying the two-sided regulator to X^* . If $y = (y_t, t \geq 0)$ is an element of C with $0 \leq y_0 \leq b$, let us set $\Lambda(y) \equiv K(f(y), g(y), h(y))$. Because $L = f(X)$, $U = g(X)$, and $Z = h(X)$, we have

$$(11) \quad k(x) \equiv E_x[\Lambda(X)].$$

Similarly, (10) implies that

$$(12) \quad K(L^*, U^*, Z^*) = \Lambda(X^*),$$

and, of course, $Z_T = X_0^*$, so the proposition will be proved if we can establish that

$$(13) \quad E_x[\Lambda(X^*) | \mathcal{F}_T] = k(X_0^*) .$$

Now recall the strong Markov property (3.0.5) of X . From (1.4.1) we have that $X_{T+t} - X_T$ is independent of $\mathcal{F}_T$. Because $Z_T \in \mathcal{F}_T$, it follows that (3.0.5) continues to hold when X^* is defined by (9), although a different definition was used in §3.0. Equation (13) then follows immediately, because $k(x) \equiv E_x[\Lambda(X)]$. $\square$

In the remainder of this chapter there will be no need to mention explicitly the mapping that carries X into (L, U, Z) ; we use only the fact that X , L , U , and Z together satisfy (1) to (3). Thus the letters f , g , and h , can and will be reused with new meanings.

§2. APPLICATION OF ITO'S FORMULA

Fixing $x \in [0, b]$ throughout this section, we set

$$W_t \equiv \frac{1}{\sigma}(X_t - X_0 - \mu t), \quad t \geq 0 .$$

Then W is a standard Brownian motion on the filtered probability space $(\Omega, \mathbb{F}, P_x)$ and $Z \equiv X + L - U$ is an Ito process with Brownian component σW and VF component $\mu t + L - U$. Let $f: [0, b] \rightarrow R$ be twice continuously differentiable. As in Chapters 3 and 4 define the differential operator Γ via

$$\Gamma f \equiv \frac{1}{2} \sigma^2 f'' + \mu f' .$$

(1) Proposition. $f(Z)$ is an Ito process with differential

$$df(Z) = \sigma f'(Z) dW + [\Gamma f(Z) dt + f'(0) dL - f'(b) dU] .$$

Remark. The Brownian component of $f(Z)$ has differential $\sigma f'(Z) dW$, whereas the quantity in square brackets is the differential of the VF component. Note that the coefficients of dL and dU are constants.

Proof. Proceeding exactly as in the proof of (4.9.1), we apply Ito's formula to deduce that

$$(2) \quad \begin{aligned} df(Z) &= f'(Z) dZ + \frac{1}{2} f''(Z) (dZ)^2 \\ &= f'(Z) (dX + dL - dU) + \frac{1}{2} f''(Z) \sigma^2 dt \end{aligned}$$

$$\begin{aligned} &= f'(Z) (\sigma dW + \mu dt + dL - dU) + \frac{1}{2} \sigma^2 f''(Z) dt \\ &= \sigma f'(Z) dW + \Gamma f(Z) dt + f'(Z) dL - f'(Z) dU . \end{aligned}$$

In its exact integral form, (2) says that

$$(3) \quad \begin{aligned} f(Z_t) &= f(Z_0) + \sigma \int_0^t f'(Z) dW + \int_0^t \Gamma f(Z) ds \\ &\quad + \int_0^t f'(Z) dL - \int_0^t f'(Z) dU . \end{aligned}$$

But (1.3) gives

$$\int_0^t f'(Z) dL = \int_0^t f'(0) dL = f'(0) L_t ,$$

and similarly for $\int f'(Z) dU$. Making these substitutions in (3) completes the proof. $\square$

(4) Corollary. Given $\lambda > 0$, set $u \equiv \lambda f - \Gamma f$ on $[0, b]$. Also, let $c \equiv f'(0)$ and $r \equiv f'(b)$. Then

$$(5) \quad f(x) = E_x \left\{ \int_0^\infty e^{-\lambda t} [u(Z) dt - c dL + r dU] \right\} .$$

Proof. Proceeding exactly as in the proof of (4.9.2), we use the specialized integration by parts formula (4.8.6) and then Proposition (1) to obtain

$$(6) \quad \begin{aligned} e^{-\lambda t} f(Z_t) &= f(Z_0) + \int_0^t e^{-\lambda s} df(Z) - \lambda \int_0^t e^{-\lambda s} f(Z) ds \\ &= f(Z_0) + M_t - \int_0^t e^{-\lambda s} [u(Z) ds - c dL + r dU] , \end{aligned}$$

where

$$(7) \quad M_t \equiv \sigma \int_0^t e^{-\lambda s} f'(Z) dW .$$

The integrand on the right side of (7) is bounded because $0 \leq Z \leq b$, so $E_x(M_t) = 0$ by (4.3.7). Also, $\exp(-\lambda t)f(Z_t) \rightarrow 0$ as $t \rightarrow \infty$ because $f(Z)$ is bounded and thus (5) is obtained by taking E_x of both sides in (6) and letting $t \rightarrow \infty$. $\square$

§3. EXPECTED DISCOUNTED COSTS

Hereafter let $\lambda > 0$ be a fixed interest rate. Given a continuous cost rate function u on $[0, b]$ and real constants c and r , we wish to calculate

$$(1) \quad k(x) \equiv E_x \left\{ \int_0^\infty e^{-\lambda t} [u(Z) dt - c dL + r dU] \right\}.$$

For motivation of this problem, see §2.5 and Chapter 6. Corollary (2.4) shows that to compute k one need only solve the ordinary differential equation

$$(2) \quad \lambda k(x) - \Gamma k(x) = u(x), \quad 0 \leq x \leq b$$

with boundary conditions

$$(3) \quad k'(0) = c \quad \text{and} \quad k'(b) = r.$$

Rather than attacking this analytical problem directly, we first use the strong Markov property of (L, U, Z) to obtain a partial solution by probabilistic reasoning. Let $T \equiv T(0) \wedge T(b)$ and define

$$(4) \quad h(x) \equiv E_x \left\{ \int_0^T e^{-\lambda t} u(Z) dt \right\}, \quad 0 \leq x \leq b.$$

In §3.5 we derived a general formula for h in terms of u , observing afterward that

$$(5) \quad \lambda h(x) - \Gamma h(x) = u(x), \quad 0 \leq x \leq b,$$

and

$$(6) \quad h(0) = h(b) = 0.$$

(7) Proposition. Let $\psi_*(x)$ and $\psi^*(x)$ be defined on $[0, b]$ as in §3.2. Then

$$(8) \quad k(x) = h(x) + \psi_*(x)k(0) + \psi^*(x)k(b), \quad 0 \leq x \leq b.$$

Proof. We shall apply the strong Markov property (1.8) using the particular functional

$$(9) \quad K(L, U, Z) = \int_0^\infty e^{-\lambda t} [u(Z) dt + c dL - r dU].$$

Taking E_x of both sides in (9), it is seen that the definitions of k advanced in §1 and this section agree. Comparing (1) and (4) and using the fact that $L_T = U_T = 0$, we have

$$(10) \quad k(x) = h(x) + E_x \left\{ \int_T^\infty e^{-\lambda t} [u(Z) dt + c dL - r dU] \right\}.$$

Let L^* , U^* , and Z^* be defined as in §1. Proceeding as in (3.4.9), but using (1.8) rather than the strong Markov property of X , one finds that

$$\begin{aligned} & E_x \left\{ \int_T^\infty e^{-\lambda t} [u(Z) dt + c dL - r dU] \right\} \\ &= E_x \left\{ e^{-\lambda T} \int_0^\infty e^{-\lambda t} [u(Z^*) dt + c dL^* - r dU^*] \right\} \\ &= E_x \{ e^{-\lambda T} K(L^*, U^*, Z^*) \} \\ &= E_x \{ e^{-\lambda T} E_x [K(L^*, U^*, Z^*) | \mathcal{F}_T] \} \\ &= E_x \{ e^{-\lambda T} k(Z_T) \} \\ &= k(0) E_x(e^{-\lambda T}; Z_T = 0) + k(b) E_x(e^{-\lambda T}; Z_T = b) \\ &= k(0)\psi_*(x) + k(b)\psi^*(x). \end{aligned}$$

Combining this with (10) proves the desired identity. $\square$

Explicit formulas for $h(x)$, $\psi_*(x)$, and $\psi^*(x)$ have been derived in Chapter 3, so equation (8) reduces our problem to determination of the constants $k(0)$ and $k(b)$. Recall from §3.2 that ψ_* and ψ^* both satisfy $\lambda\psi - \Gamma\psi = 0$. Thus any function k of the general form (8) will satisfy the main equation (2), and one simply chooses $k(0)$ and $k(b)$ so as to satisfy the boundary conditions (3). An examination of the solutions derived in Chapter 3 will show that (8) is equivalent to the general form

$$(11) \quad k(x) = f(x) + Ae^{-\alpha_*(\lambda)x} + Be^{\alpha^*(\lambda)x},$$

where $\alpha_*(\lambda)$ and $\alpha^*(\lambda)$ are the constants defined by (3.2.14) and (3.2.15), respectively, A and B are constants to be determined, and

$$(12) \quad f(x) \equiv E_x \left\{ \int_0^\infty e^{-\lambda t} u(X_t) dt \right\}.$$

In this chapter we have treated u as a function on $[0, b]$; one may use any convenient extension of u for purposes of (12). Again it can be verified that any k of the form (11) satisfies the main equation (2), therefore one must select A and B so as to meet the boundary conditions (3).

§4. REGENERATIVE STRUCTURE

Let the starting state $Z(0) = X(0) = x \in [0, b]$ be fixed throughout this section, so we are working with a single filtered probability space $(\Omega, \mathbb{F}, P_x)$. Let

$$(1) \quad T_0 \equiv \inf \{t \geq 0 : Z(t) = 0\}$$

and then for $n = 0, 1, 2, \dots$ inductively define

$$(2) \quad Z_{n+1}^*(t) \equiv Z(T_n + t), \quad t \geq 0,$$

$$(3) \quad L_{n+1}^*(t) \equiv L(T_n + t) - L(T_n), \quad t \geq 0,$$

$$(4) \quad U_{n+1}^*(t) \equiv U(T_n + t) - U(T_n), \quad t \geq 0,$$

and

$$(5) \quad T_{n+1} \equiv \text{smallest } t > T_n \text{ such that } Z(t) = 0 \text{ and}$$

$$Z(s) = b \text{ for some } s \in (T_n, t).$$

In words, T_0 is the first hitting time of level zero, and T_{n+1} is the first time after T_n at which Z returns to level zero after first visiting level b . Then $T_0, T_1, \dots$ are stopping times, and it follows directly from Proposition (1.7) that, for any $n = 1, 2, \dots$ and any bounded, measurable $K: C \times C \times C \rightarrow R$,

$$(6) \quad E_x[K(L_n^*, U_n^*, Z_n^*) | \mathcal{F}(T_{n-1})] = E_0[K(L, U, Z)].$$

Let $\tau_n \equiv T_n - T_{n-1}$ for $n = 1, 2, \dots$, and set $\tau \equiv \tau_1$ for ease of notation. It follows from (6) that

$$(7) \quad \{\tau_1, \tau_2, \dots\} \text{ are IID random variables,}$$

and it is left as an exercise (see Problem 7) to show that

$$(8) \quad \{\tau_1, \tau_2, \dots\} \text{ have a nonlattice (or aperiodic or nonarithmetic) distribution with } E_x(\tau_n) = E_0(\tau) < \infty.$$

Conditions (6) to (8) describe the regenerative structure of our Brownian flow system (L, U, Z) . After an initial delay of duration T_0 , the *regeneration times* $T_1, T_2, \dots$ divide the evolution of (L, U, Z) into independent and identically distributed blocks (or regenerative cycles) of duration $\tau_1, \tau_2, \dots$. Specifically, it follows from (6) that

$$(9) \quad \{L_1^*(\tau_1), L_2^*(\tau_2), \dots\} \text{ and } \{U_1^*(\tau_1), U_2^*(\tau_2), \dots\} \text{ are IID sequences and their distributions do not depend on } x.$$

We now define

$$(10) \quad \alpha \equiv \frac{E_0[L(\tau)]}{E_0(\tau)},$$

$$(11) \quad \beta \equiv \frac{E_0[U(\tau)]}{E_0(\tau)},$$

$$(12) \quad \pi(A) \equiv \frac{E_0\{\int_0^\tau 1_A(Z_t) dt\}}{E_0(\tau)}$$

for Borel subsets A of $[0, b]$. One might describe α and β as the expected increase per unit time in L and U , respectively, over a regenerative cycle. Similarly, $\pi(A)$ is the expected amount of time Z spends in the set A during a regenerative cycle, normalized to make $\pi(\cdot)$ a probability measure. The following proposition is a standard application of renewal theory, so the proof will only be sketched. See Section §9.2 of Çinlar (1975) for a similar analysis of regenerative processes.

(13) Proposition. Let A be an interval subset of $[0, b]$. Then

$$(14) \quad P_x\{Z(t) \in A\} \rightarrow \pi(A) \quad \text{as } t \rightarrow \infty,$$

$$(15) \quad \frac{1}{t} E_x[L(t)] \rightarrow \alpha \quad \text{as } t \rightarrow \infty,$$

and

$$(16) \quad \frac{1}{t} E_x[U(t)] \rightarrow \beta \quad \text{as } t \rightarrow \infty.$$

Remark. Thus π , originally defined as an expected occupancy measure during a regenerative cycle, is also the limit distribution of Z , regardless of starting state. In the problems at the end of this chapter, it will be seen that π is also the unique stationary distribution of the Markov process Z and that it may be viewed as a long-run average occupancy distribution. For each of these interpretations of π , there is a corresponding interpretation of α and β , as the problems will show.

Proof. For simplicity, let us assume that $x = 0$, in which case $T_0 = 0$ (the first regenerative cycle begins immediately). To simplify typography, we write $P(\cdot)$ in place of $P_0(\cdot)$. Let $F(t) \equiv P\{\tau \leq t\}$ for $t \geq 0$, noting that F is a nonlattice distribution with

$$a \equiv E(\tau) = \int_0^\infty t F(dt) < \infty$$

by (8). First, we have the obvious decomposition

$$(17) \quad P\{Z(t) \in A\} = P\{Z(t) \in A, \tau > t\} + \int_0^t P\{\tau \in ds, Z(\tau + t - s) \in A\}.$$

From the key condition (6) one deduces that

$$(18) \quad \begin{aligned} \int_0^t P\{\tau \in ds, Z(\tau + t - s) \in A\} &= \int_0^t P\{\tau \in ds\} P\{Z(t - s) \in A\} \\ &= \int_0^t P\{Z(t - s) \in A\} F(ds). \end{aligned}$$

Second, from (17), (18), and the key renewal theorem (cf. pp. 294–295 of Çinlar, 1975), it follows that

$$(19) \quad P\{Z(t) \in A\} \rightarrow \frac{1}{a} \int_0^\infty P\{Z(t) \in A, T > t\} dt.$$

Finally, to deduce (14) from (19), we use Fubini's theorem (see §A.5) to write

$$\begin{aligned} \int_0^\infty P\{Z(t) \in A, \tau > t\} dt &= \int_0^\infty E[1_{\{Z(t) \in A, \tau > t\}}] dt \\ &= E \left[\int_0^\infty 1_{\{Z(t) \in A\}} 1_{\{\tau > t\}} dt \right] \\ &= E \left[\int_0^\tau 1_{\{Z(t) \in A\}} dt \right] = a \pi(A). \end{aligned}$$

To prove (15), first set $Y_n \equiv L_n^*(\tau_n)$ and $S_n \equiv Y_1 + \cdots + Y_n$ for $n = 1, 2, \dots$, with $S_0 \equiv 0$. Also, let $N(t) \equiv \sup\{n : T_n \leq t\}$ for $t \geq 0$, so $N \equiv \{N(t), t \geq 0\}$ is a renewal process with interarrival times $\tau_1, \tau_2, \dots$. The key observation is that

$$(20) \quad S_{N(t)} \leq L(t) \leq S_{N(t)+1} \quad \text{for } t \geq 0,$$

and thus

$$(21) \quad \frac{1}{t} E[S_{N(t)}] \leq \frac{1}{t} E[L(t)] \leq \frac{1}{t} E[S_{N(t)+1}].$$

The argument on pp. 78–79 of Ross (1983), using Wald's identity and the elementary renewal theorem, shows that the upper and the lower bounds in (21) both approach $E(Y_1)/E(\tau_1)$ as $t \rightarrow \infty$. Thus $E[L(t)]/t \rightarrow E(Y_1)/E(\tau_1)$ as $t \rightarrow \infty$, which is precisely (15), and (16) is established similarly. $\square$

§5. THE STEADY-STATE DISTRIBUTION

We now derive a useful relationship, based on Ito's Formula, from which one can compute the steady-state quantities $\pi(\cdot)$, α , and β . Let the initial state be $x = 0$, so we are working with the filtered probability space $(\Omega, \mathbb{F}, P_0)$. Proposition (2.1) gives

$$(1) \quad \begin{aligned} f(Z_t) &= f(Z_0) + \sigma \int_0^t f'(Z) dW \\ &\quad + \int_0^t \Gamma f(Z) ds + f'(0)L_t - f'(b)U_t \end{aligned}$$

for any $f: R \rightarrow R$ that is twice continuously differentiable. Substituting τ for t in (1), we see that $f(Z_\tau) = f(Z_0) = f(0)$. Now take E_0 of both sides. The

Ito integral on the right side has expected value zero by (4.3.7) because the integrand is bounded and $E(\tau) < \infty$. Thus

$$(2) \quad 0 = E_0 \left\{ \int_0^\tau \Gamma f(Z) dt \right\} + f'(0)E_0(L_\tau) - f'(b)E_0(U_\tau) .$$

Furthermore, from the definition (4.12) of π , it follows that

$$(3) \quad E_0 \left\{ \int_0^\tau \Gamma f(Z) dt \right\} = E_0(\tau) \int_{[0,b]} \Gamma f(z) \pi(dz);$$

this relationship holds by definition if Γf is the indicator of a set, then by linearity it holds whenever Γf is a simple function (linear combination of indicators), and then by monotone convergence it holds in general. Finally, $E_0(L_\tau) = \alpha E_0(\tau)$ and $E_0(U_\tau) = \beta E_0(\tau)$ by definition. Substituting these identities and (3) into (2), then dividing by $E_0(\tau)$, one arrives at the key relationship

$$(4) \quad 0 = \int_{[0,b]} \Gamma f(z) \pi(dz) + \alpha f'(0) - \beta f'(b) .$$

(5) Proposition. If $\mu = 0$, then $\alpha = \beta = \sigma^2/2b$ and π is the uniform distribution on $[0,b]$. Otherwise, setting $\theta \equiv 2\mu/\sigma^2$,

$$(6) \quad \alpha = \frac{\mu}{e^{\theta b} - 1}, \quad \beta = \frac{\mu}{1 - e^{-\theta b}},$$

and π is the truncated exponential distribution

$$(7) \quad \mu(dz) = p(z) dz, \quad \text{where} \quad p(z) = \frac{\theta e^{\theta z}}{e^{\theta b} - 1} .$$

Proof. First suppose $\mu = 0$. Substitute in (4) the linear function $f(z) = z$. Then $\Gamma f = 0$, $f'(0) = f'(b) = 1$, and (4) gives $\alpha = \beta$. Next take $f(z) = z^2$, so that $\Gamma f(z) = \sigma^2$, $f'(0) = 0$, and $f'(b) = 2b$. Then (4) yields $\alpha = \beta = \sigma^2/2b$. Finally, consider the exponential function $f(z) = \exp(\lambda z)$. Substituting this in (4) and using the known values of α and β , we arrive at

$$\int_{[0,b]} e^{\lambda y} \pi(dy) = \frac{1}{b\lambda} (e^{b\lambda} - 1), \quad \lambda \in \mathbb{R} ,$$

which is the transform of the uniform distribution as desired. If μ is nonzero, substitution of the test function $f(z) = z$ in (4) gives

$$(8) \quad 0 = \mu + \alpha - \beta .$$

Now consider again the exponential test function $f(z) = \exp(\lambda z)$ so that

$$(9) \quad \Gamma f(z) = (\tfrac{1}{2} \sigma^2 \lambda^2 + \mu \lambda) e^{\lambda z}$$

$$(10) \quad f'(0) = \lambda \quad \text{and} \quad f'(b) = \lambda e^{\lambda b} .$$

By taking $\lambda = -2\mu/\sigma^2 = -\theta$, we have $\Gamma f = 0$, and hence (4) yields

$$(11) \quad 0 = -\theta \alpha + \theta \beta e^{-\theta b} .$$

Solving (8) and (11) simultaneously gives (6). Now let us return to general λ . Using (9), (10), and (6) in (4), we arrive at

$$\int_{[0,b]} e^{\lambda z} \pi(dz) = \left(\frac{\theta}{\theta + \lambda} \right) \left[\frac{e^{(\theta + \lambda)b} - 1}{e^{\theta b} - 1} \right] ,$$

which is the transform of the truncated exponential distribution (7), as desired. $\square$

An important quantity in applications is the mean of the steady-state distribution (7). Readers may verify that

$$(12) \quad \gamma \equiv \int_0^b z p(z) dz = \frac{b}{1 - e^{-\theta b}} - \frac{1}{\theta} .$$

To express the system performance measures α , β , γ in more compact form, let

$$(13) \quad \psi(\xi) \equiv \frac{\xi}{e^\xi - 1}$$

and

$$(14) \quad \phi(\xi) \equiv \frac{e^{-\xi} - 1 + \xi}{\xi(1 - e^{-\xi})}$$

with $\psi(0) = 1$ and $\phi(0) = \frac{1}{2}$ (these are the values that make ψ and ϕ continuous at the origin). It has been shown in this section that

$$(15) \quad \alpha = \left(\frac{\sigma^2}{2b} \right) \psi \left(\frac{2\mu b}{\sigma^2} \right),$$

$$(16) \quad \beta = \alpha + \mu,$$

and

$$(17) \quad \gamma = b \phi \left(\frac{2\mu b}{\sigma^2} \right).$$

Incidentally, it follows from (4.14) that

$$(18) \quad E_x(Z_t) \rightarrow \gamma \quad \text{as } t \rightarrow \infty.$$

§6. THE CASE OF A SINGLE BARRIER

Assuming that $X_0 = x \geq 0$, let us now consider the processes (L, Z) obtained by applying to X the one-sided regulator of §2.2. (Recall that the distribution of Z_t was calculated explicitly for general values of t and x in §3.6.) Each of the results developed in §1 to 5 has a precise analog in the case of a single barrier, and the most important of these will be recorded here with the proofs left as exercises. Recall from §2.2 that

- (1) L is increasing and continuous with $L_0 = 0$,
- (2) $Z_t \equiv X_t + L_t \geq 0$ for all $t \geq 0$, and
- (3) L increases only when $Z = 0$.

Thus Z is an Ito process with Brownian component σW and VF component $\mu t + L$. Using (3) and Ito's formula, one finds that

$$(4) \quad f(Z_t) = f(Z_0) + \sigma \int_0^t f'(Z) dW + \int_0^t \Gamma f(Z) dt + f'(0)L_t$$

for any $f: R \rightarrow R$ that is twice continuously differentiable. If f also has bounded derivative, then it follows from (4) that, for any $\lambda > 0$,

$$(5) \quad f(x) = E_x \left\{ \int_0^\infty e^{-\lambda t} [u(Z) dt - c dL] \right\},$$

where

$$(6) \quad u(x) \equiv \lambda f(x) - \Gamma f(x) \quad \text{and} \quad c \equiv f'(0).$$

Turning this calculation around, suppose there is given a constant c and a bounded, continuous cost rate function $u: [0, \infty) \rightarrow R$. If we wish to calculate the expected discounted cost

$$(7) \quad k(x) \equiv E_x \left\{ \int_0^\infty e^{-\lambda t} [u(Z) dt - c dL] \right\}, \quad x \geq 0,$$

it suffices to solve the differential equation $\lambda k(x) - \Gamma k(x) = u(x)$, $x \geq 0$, subject to the requirement that $k'(0) = c$ and $k'(\cdot)$ is bounded on $[0, \infty)$. Imitating the arguments in §1 and §3, it can be shown that

$$(8) \quad k(x) = g(x) + k(0) e^{-\alpha_*(\lambda)x}, \quad x \geq 0,$$

where

$$(9) \quad g(x) \equiv E_x \left\{ \int_0^T e^{-\lambda t} u(X_t) dt \right\}, \quad x \geq 0,$$

and $T = \inf\{t \geq 0 : X_t = 0\}$. A general formula for g was derived in §3.4, and it follows from the results of §3.2 to 3.4 that any function k of the form (8) satisfies $\lambda k - \Gamma k = u$ on $[0, \infty)$. Moreover, boundedness of u implies boundedness of g' , and the boundary condition $k'(0) = c$ can be satisfied by taking

$$(10) \quad k(0) = \frac{g'(0) - c}{\alpha_*(\lambda)};$$

thus formulas (8) to (10) provide a complete solution of the problem at hand. For simplicity, this treatment of expected discounted costs has been restricted to bounded cost rate functions u . But the solution (8) to (10) remains valid so long as the expectation in (9) makes sense, as one can show with a truncation argument.

Asymptotic analysis of Z is much easier with one barrier than with two,

because we have previously calculated the distribution of Z_t for finite t . Specifically, letting $t \rightarrow \infty$ in formula (3.6.1), one finds that (for any $z \geq 0$)

$$(11) \quad P_x\{Z_t \leq z\} \rightarrow 1 - \exp\left(\frac{2\mu z}{\sigma^2}\right) \quad \text{if } \mu < 0,$$

whereas $P_x\{Z_t \leq z\} \rightarrow 0$ if $\mu \geq 0$ as one would expect. Note that the exponential limit distribution in (11) is what one gets by simply letting $b \rightarrow \infty$ in the steady-state distribution calculated earlier in §5. From formula (3.6.1) it also follows that

$$(12) \quad E_x(Z_t) \rightarrow \frac{\sigma^2}{2|\mu|} \quad \text{as } t \rightarrow \infty \quad \text{if } \mu < 0,$$

which is what one would expect from (11). Using the fact that $E_x(X_t) = x + \mu t$, we can take E_x of both sides in (2) to obtain

$$(13) \quad E_x(Z_t) = x + \mu t + E_x(L_t).$$

Now divide (13) by t , let $t \rightarrow \infty$, and use (12) to conclude that

$$(14) \quad \frac{1}{t} E_x(L_t) \rightarrow |\mu| \quad \text{as } t \rightarrow \infty \quad \text{if } \mu < 0.$$

Readers should note that the constant α computed earlier in §5 approaches $|\mu|$ as $b \rightarrow \infty$ if $\mu < 0$, which is consistent with (14).

PROBLEMS AND COMPLEMENTS

1. In the setting of §2, let $T \equiv \inf\{t \geq 0 : Z_t = b\}$ and note that $U(T) = 0$. Let $f: R \rightarrow R$ be twice continuously differentiable. Use (2.1) to prove that

$$E_x[f(Z(t \wedge T))] = f(x) + E_x\left[\int_0^{t \wedge T} \Gamma f(Z) ds\right] + f'(0) E_x[L(t \wedge T)].$$

Specializing to $f(x) \equiv \exp(\lambda x)$, note that $f'(0) = \lambda$, and $\Gamma f(x) = q(\lambda)f(x)$, where $q(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \mu\lambda$. Choosing $\lambda > 0$ large enough to ensure $q(\lambda) > 0$, show that

$$(*) \quad E_x[f(Z(t \wedge T))] \geq f(x) + q(\lambda) E_x(t \wedge T) + \lambda E_x[L(t \wedge T)].$$

But $f(Z(t \wedge T)) \leq f(b)$, and $L(\cdot) \geq 0$, so $E_x(t \wedge T) \leq [f(b) - f(x)]/q(\lambda)$ by (*). Now let $t \uparrow \infty$ and use the monotone convergence theorem to conclude that $E_x(T) < \infty$, $0 \leq x \leq b$.

2. **(Continuation)** Let f again be general. Use (2.1) and (4.3.7) to show that

$$f(b) = f(x) + E_x\left[\int_0^T \Gamma f(Z) dt\right] + f'(0)E_x[L(T)], \quad 0 \leq x \leq b.$$

3. **(Continuation)** Fix $\lambda > 0$ and define $\phi(x) \equiv E_x[\exp(-\lambda T)]$. Show that

$$f(b)\phi(x) = f(x) + E_x\left\{\int_0^T e^{-\lambda t}[(\Gamma f - \lambda f)(Z) dt + f'(0) dL]\right\}.$$

4. **(Continuation)** Let $\alpha_*(\lambda)$ and $\alpha^*(\lambda)$ be defined as in §3.2 and set

$$g(x) \equiv \alpha^*(\lambda)e^{-\alpha^*(\lambda)x} + \alpha_*(\lambda)e^{\alpha^*(\lambda)x}, \quad 0 \leq x \leq b.$$

From the results of §3.2 it follows that $\lambda g - \Gamma g = 0$ on $[0, b]$ and clearly $g'(0) = 0$. Conclude that $\phi(x) = g(x)/g(b)$, $0 \leq x \leq b$.

5. Again consider the setup of §2. Fix $\lambda > 0$, let $f: [0, b] \rightarrow R$ be twice continuously differentiable, and let

$$V_t \equiv e^{-\lambda L_t} f(Z_t), \quad t \geq 0.$$

Use the integration by parts formula (4.8.5) to calculate the differential of the Ito process V .

6. **(Continuation)** Now let $f(x) \equiv E_x\{\int_0^T e^{-\lambda L_t} u(Z_t) dt\}$, $0 \leq x \leq b$, where u is continuous on $[0, b]$ and T is the first hitting time of b as in Problems 1 to 4. Show that to compute f it suffices to solve the differential equation

$$\Gamma f + u = 0$$

with boundary conditions

$$f'(0) - \lambda f(0) = 0 \quad \text{and} \quad f(b) = 0.$$

7. In the setting of §4, let $T(y) \equiv \inf\{t \geq 0 : Z_t = y\}$. In Problem 1 it was shown that $E_x[T(b)] < \infty$, $0 \leq x \leq b$, and essentially the same argument gives $E_x[T(0)] < \infty$. Use Proposition (1.8) to show that $E_0(\tau) = E_0[T(b)] + E_b[T(0)] < \infty$.
8. In the setting of §4 (where the starting state x is viewed as a fixed constant), it can be shown that

$$\frac{1}{t} \int_0^t 1_A(Z) ds \rightarrow \pi(A) \quad \text{almost surely as } t \rightarrow \infty,$$

$$\frac{1}{t} L(t) \rightarrow \alpha \quad \text{almost surely as } t \rightarrow \infty,$$

and

$$\frac{1}{t} U(t) \rightarrow \beta \quad \text{almost surely as } t \rightarrow \infty.$$

For this one uses the regenerative structure of (L, U, Z) , the standard form of the strong law of large numbers and the strong law for renewal processes. A very similar argument can be found on page 78 of Ross (1983).

9. A probability measure π on $[0, b]$ is said to be a stationary distribution for Z if

$$(*) \quad \int_{[0, b]} \pi(dx) E_x[f(Z_t)] = \int_{[0, b]} \pi(dz) f(z)$$

for all $t \geq 0$ and all bounded, measurable f . Condition $(*)$ says that if the initial state of Z is randomized with distribution π , then Z_t has distribution π at each future time t . Use (1.8) to show that if π is a stationary distribution for Z , then

$$\int_{[0, b]} \pi(dx) E_x(L_t) = \alpha t \quad \text{and} \quad \int_{[0, b]} \pi(dx) E_x(U_t) = \beta t$$

for all $t \geq 0$, where α and β are constants yet to be determined.

10. **(Continuation)** Let f be bounded and measurable on $[0, b]$. One immediate consequence of (1.8) is that

$$E_x[f(Z_{t+s})] = \int_{[0, b]} P_x\{Z_t \in dy\} E_y[f(Z_s)].$$

Letting $t \rightarrow \infty$, use the bounded convergence theorem to conclude that the limit distribution calculated in §5 is also a stationary distribution for Z . Now to prove uniqueness, let π be any stationary distribution. From (2.1) it follows that

$$E_x[f(Z_t)] = f(x) + E_x \left[\int_0^t \Gamma f(Z) ds + f'(0)L_t - f'(b)U_t \right]$$

for any $t \geq 0$, $x \in [0, b]$ and twice continuously differentiable f . Integrate both sides of this relation with respect to $\pi(dx)$, then use the identities displayed in Problem 7 to show that π , α , and β jointly satisfy (5.4) for all twice continuously differentiable test functions f . Thus π , α , and β are the same quantities computed in §5.

11. Consider the setup of §6, where $Z \equiv X + L$ is regulated (μ, σ) Brownian motion with a single barrier at zero. Let $f(t, x)$ be twice continuously differentiable on R^2 , and define

$$g(t, x) \equiv \frac{\partial}{\partial x} f(t, x)$$

and

$$h(t, x) \equiv \left(\frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) f(t, x).$$

Use the multidimensional Ito formula (4.5.4) to show that

$$f(t, Z_t) = f(0, Z_0) + \sigma \int_0^t g(s, Z_s) dW + \int_0^t h(s, Z_s) ds + \int_0^t g(s, 0) dL.$$

Now suppose that $g(s, y)$ is bounded on $[0, t] \times [0, \infty)$, that $h(s, y) = 0$ on $[0, t] \times [0, \infty)$ and that $g(s, 0) = 0$ on $[0, t]$. Use (4.3.7) to show that $f(0, x) = E_x[f(t, Z_t)]$ for $x \geq 0$.

12. **(Continuation)** Fix $y \geq 0$ and let $Q(t, x, y)$ be defined by formula (3.6.1), recalling that

$$\frac{\partial}{\partial t} Q(t, x, y) = \left(\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} \right) Q(t, x, y),$$

$$\frac{\partial}{\partial x} Q(t, 0, y) = 0, \quad \text{and} \quad Q(0, x, y) = 1_{(x > y)}.$$

It is also easy to check that $(\partial/\partial x)Q(t, x, y)$ is bounded as a function of t and x . Use the result of Problem 11 to prove $Q(t, x, y) = P_x\{Z_t > y\}$, thus verifying the interpretation of Q given in §3.6. This requires a sequence of steps exactly like those outlined in Problems 9 to 12 of Chapter 4.

13. It is the purpose of this problem to give some idea of the role played by stochastic calculus in the analysis of multidimensional Brownian flow systems. Consider the three-stage flow system, or tandem storage system, discussed earlier in Problem 2.3. Suppose that the netput process $X = (X_1, X_2)$ is modeled as a two-dimensional standard Brownian motion. (This means that X_1 and X_2 are independent, each with zero drift and unit variance. Similar results are obtained with arbitrary drift vector and covariance matrix.) Let S (for *state space*) denote the positive quadrant of R^2 . We extend our previous notational system to denote by P_x the probability measure on the path space of X corresponding to starting state $x = (x_1, x_2) \in S$. Applying to X the multidimensional regulator of Problem 2.3, one obtains processes $L = (L_1, L_2)$ and $Z = (Z_1, Z_2)$ satisfying

- (1) L_1 and L_2 are increasing and continuous with $L_1(0) = L_2(0) = 0$,
- (2) $Z_1(t) \equiv X_1(t) + L_1(t) \geq 0$ for all $t \geq 0$,
- (3) $Z_2(t) \equiv X_2(t) - L_1(t) + L_2(t) \geq 0$ for all $t \geq 0$, and
- (4) L_1 and L_2 increase only when $Z_1 = 0$ and $Z_2 = 0$, respectively.

Recall that the path-to-path mapping that carries X into (L, Z) is naturally described by the *directions of control* shown in Figure 1. From (1) to (3) we see that Z is a two-dimensional Ito process. Now let $f: R^2 \rightarrow R$ be twice continuously differentiable and define the differential operators (here subscripts denote partial derivatives as in §4.5)

$$(5) \quad \Delta f \equiv f_{11} + f_{22},$$

$$(6) \quad D_1 f \equiv f_1 - f_2,$$

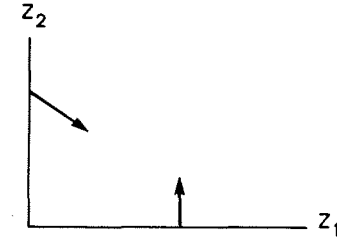


Figure 1. Directions of control for Z .

and

$$(7) \quad D_2 f \equiv f_2.$$

Note that D_1 and D_2 are directional derivatives for the directions of control associated with the boundary surfaces $Z_1 = 0$ and $Z_2 = 0$, respectively. Apply the multidimensional Ito formula to show that

$$(8) \quad df(Z) = \sum_{i=1}^2 f_i(Z) dZ_i + \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 f_{ij}(Z) dZ_i dZ_j$$

$$= \sum_{i=1}^2 f_i(Z) dX_i + \frac{1}{2} \Delta f(Z) dt + \sum_{i=1}^2 D_i f(Z) dL_i.$$

This provides a precise analog for the basic relationship (2.1) characterizing one-dimensional regulated Brownian motion. Now proceeding as in §2, we can use (8) and the specialized integration by parts formula (4.8.6) to obtain

$$(9) \quad e^{-\lambda T} f(Z(T)) = f(Z(0)) + \sum_{i=1}^2 \int_0^T e^{-\lambda t} f_i(Z) dX_i$$

$$+ \int_0^T e^{-\lambda t} \left(\frac{1}{2} \Delta f - \lambda f \right)(Z) dt$$

$$+ \sum_{i=1}^2 \int_0^T e^{-\lambda t} D_i f(Z) dL_i$$

for any constant $\lambda > 0$ and stopping time $T < \infty$. Let T be a fixed time, suppose f and its first-order partials are bounded on S , and take E_x of both sides in (9). The stochastic integrals have expected value zero by (4.3.7), and upon letting $T \rightarrow \infty$ we arrive at

$$(10) \quad f(x) = E_x \left\{ \int_0^\infty e^{-\lambda t} \left[\left(\lambda f - \frac{1}{2} \Delta f \right)(Z) dt - \sum_{i=1}^2 Df_i(Z) dL_i \right] \right\}.$$

Given constants c_1, c_2 , and a well behaved cost rate function $u: S \rightarrow R$, suppose we wish to calculate

$$(11) \quad k(x) \equiv E_x \left\{ \int_0^\infty e^{-\lambda t} [u(Z) dt - c_1 dL_1 - c_2 dL_2] \right\}.$$

Using (10), show that it suffices to find a sufficiently regular function k satisfying the partial differential equation

$$(12) \quad \frac{1}{2} \Delta k(x) - \lambda k(x) + u(x) = 0, \quad x \in S,$$

subject to the boundary conditions

$$(13) \quad D_1 k(0, x_2) = c_1, \quad x_2 \geq 0$$

and

$$(14) \quad D_2 k(x_1, 0) = c_2, \quad x_1 \geq 0.$$

Justification of the boundary conditions depends critically on the sample path property (4). For more on the theory of multidimensional regulated Brownian motion, see Harrison–Reiman (1981).

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CHAPTER 6

Optimal Control of Brownian Motion

In a stochastic control problem, one observes and then seeks to favorably influence the performance of some stochastic system. Such problems involve *dynamic* optimization, meaning that observations and actions are spread out in time. This chapter is devoted to a simple but fundamental problem of linear stochastic control. We shall solve it directly from first principles, relying heavily on the ubiquitous Ito formula. It will be found that the optimal policy involves imposition of control barriers, and the parameters of that policy will be calculated explicitly.

Informally, the problem may be stated as follows. We consider a controller who continuously monitors the content of a storage system such as an inventory or a bank account. In the absence of control, the contents process $Z = \{Z_t, t \geq 0\}$ fluctuates as a (μ, σ) Brownian motion. The controller can at any time increase or decrease the content of the system by any amount desired but is obliged to keep $Z_t \geq 0$; there are also three types of costs to be considered. First, to increase the content of the system, one incurs a transaction cost of α times the size of the increase. Similarly, to decrease the content costs β times the size of the decrease. Finally, holding costs are continuously incurred at rate hZ_t . Thus we have both *linear holding costs* and *linear costs of control*. An important feature of this problem is that the controller can *instantaneously* change the content (or state) of the system. If, in contrast, increases and decreases had to be effected at finite, bounded rates, then there would be no available policies under which $Z_t \geq 0$ almost surely for all t .

The precise mathematical statement of this problem, which involves some subtlety, is presented in the first section. Later sections are devoted to solution of the problem, after which we discuss an important application to

cash management. Another application, foreshadowed by §2.5, will be discussed in the next chapter.

§1. PROBLEM FORMULATION

The data for our problem are a drift rate μ , a variance parameter $\sigma^2 > 0$, control cost parameters α and β , an interest rate $\lambda > 0$, and a holding cost rate h . It is assumed that $\alpha + \beta > 0$, for otherwise the control problem would make no sense.

We shall adopt the canonical setup of §A.3, where X is the coordinate process on C , and $\mathbb{F}$ is the filtration generated by X . Also, as in earlier chapters, let P_x be the unique probability measure on $(C, \mathcal{C})$ under which X is a (μ, σ) Brownian motion with starting state x . Attention is restricted to $x \geq 0$. A *policy* is defined as a pair of processes L and U such that

- (1) L and U are adapted and
- (2) L and U are right-continuous, increasing, and positive.

Interpret L_t as the cumulative increase in system content effected by the controller up to time t , and U_t as the corresponding cumulative decrease effected. The letters L and U were used in Chapter 5 to denote increasing processes associated with the *lower* and the *upper* boundaries, respectively. In the current context, this notation foretells the form of the optimal policy. We associate with policy (L, U) the *controlled process* $Z \equiv X + L - U$, and (L, U) is said to be *feasible* if

$$(3) \quad P_x\{Z_t \geq 0 \text{ for all } t \geq 0\} = 1 \quad \text{for all } x \geq 0,$$

$$(4) \quad E_x\left(\int_0^\infty e^{-\lambda t} dL\right) < \infty \quad \text{for all } x \geq 0,$$

and

$$(5) \quad E_x\left(\int_0^\infty e^{-\lambda t} dU\right) < \infty \quad \text{for all } x \geq 0.$$

In order to simplify discounted cost expressions, let us agree to interpret the integral in (4) as

$$(6) \quad \int_0^\infty e^{-\lambda t} dL \equiv L_0 + \int_{(0, \infty)} e^{-\lambda t} dL$$

and similarly for the integral in (5). This notational convention will be employed throughout the current chapter without further comment. We associate with a feasible policy (L, U) the *cost function*

$$(7) \quad k(x) \equiv E_x\left\{\int_0^\infty e^{-\lambda t}[hZ_t dt + \alpha dL + \beta dU]\right\}, \quad x \geq 0,$$

and (L, U) is said to be *optimal* if $k(x)$ is minimal (among the cost functions associated with feasible policies) for each $x \geq 0$.

This is the most concrete possible formulation of the control problem described informally at the beginning of the chapter. By taking $\Omega = C[0, \infty)$ and $X(\omega) = \omega$, we formally express the fact that our decisionmaker observes nothing of relevance other than the sample path of X , and (1) expresses the requirement that his or her actions over the time interval $[0, t]$ depend only on the observed values of X_s for $0 \leq s \leq t$.

(8) Proposition. Let $k(x)$ be the cost function for a feasible policy (L, U) . Also let

$$(9) \quad v(x) \equiv E_x\left\{\int_0^\infty e^{-\lambda t}(r dU - c dL)\right\}, \quad x \geq 0,$$

where $r \equiv h/\lambda - \beta$ and $c \equiv h/\lambda + \alpha$. Then

$$(10) \quad k(x) \equiv hx/\lambda + h\mu/\lambda^2 - v(x), \quad x \geq 0.$$

Remark The first two terms on the right side of (10) do not depend on the particular policy (L, U) under discussion. Thus minimization of k is equivalent to *maximization* of the *value function* v . Hereafter we shall speak in terms of the latter objective, which is easier to work with.

Proof. To simplify matters slightly, we consider only the case $L_0 = U_0 = 0$. Readers may verify that the same formula holds in general, given our notational convention (6). Because $Z \equiv X + L - U$, the first part of the expectation on the right side of (7) can be written as

$$(11) \quad hE_x\left(\int_0^\infty e^{-\lambda t} Z_t dt\right) = hE_x\left(\int_0^\infty e^{-\lambda t} X_t dt\right)$$

$$+ hE_x \left(\int_0^\infty e^{-\lambda t} (L_t - U_t) dt \right).$$

Now an application of Fubini's theorem (see §A.5) gives

$$(12) \quad E_x \left(\int_0^\infty e^{-\lambda t} X_t dt \right) = \int_0^\infty e^{-\lambda t} E_x(X_t) dt \\ = \int_0^\infty e^{-\lambda t} (x + \mu t) dt = \frac{x}{\lambda} + \frac{\mu}{\lambda^2}.$$

Next, for each fixed $T > 0$, the Riemann–Stieltjes integration by parts theorem (see §A.3) gives

$$(13) \quad \int_0^T e^{-\lambda t} dL = e^{-\lambda T} L_T + \lambda \int_0^T e^{-\lambda t} L_t dt.$$

From (4) it follows that $\exp(-\lambda T)L_T \rightarrow 0$ almost surely as $T \rightarrow \infty$. Thus letting $T \rightarrow \infty$ and then taking E_x of both sides, we obtain

$$(14) \quad E_x \left(\int_0^\infty e^{-\lambda t} L_t dt \right) = \frac{1}{\lambda} E_x \left(\int_0^\infty e^{-\lambda t} dL \right).$$

Similarly, from (5) we deduce that (14) holds with U in place of L . Equation (10) is obtained by combining these two relationships with (11), (12), and the definition (7) of k . $\square$

Hereafter, increases in L and U will be referred to as *deposits* and *withdrawals*, respectively. The maximand $v(x)$ defined by (9) may then be described as follows. Each deposit generates a *cost* of c times the deposit size, each withdrawal generates a *reward* of r times the withdrawal size, and there are no other economic considerations. We seek to maximize the expected present value of rewards received minus costs incurred over an infinite planning horizon, subject to the requirement that $Z_t \geq 0$ for all t . This maximization problem is only interesting if

$$(15) \quad 0 < r < c < \infty,$$

and it is assumed hereafter that the data satisfy (15). If the first inequality in (15) fails, then it is optimal to never make withdrawals; if the second fails, then one can make unlimited profit in a finite amount of time.

In terms of our original cost structure, h/λ is the cost of holding a unit of stock forever. Thus the cost parameter c appearing in the definition of $v(x)$ is the cost of depositing a unit of stock and then holding it in inventory forever. Similarly, r equals the infinite-horizon holding cost for a unit of stock less the transaction cost associated with a unit withdrawal.

§2. BARRIER POLICIES

To repeat, we shall hereafter seek to maximize the value function v defined by (1.9). Given the linear structure of costs and rewards, it is natural to consider the following sort of *barrier policy*. For some parameter $b > 0$, make only such withdrawals as required to keep $Z_t \leq b$, and make only such deposits as required to meet the constraint $Z_t \geq 0$. We have seen in §2.4 how such a policy can be described in precise mathematical terms. If $0 < X_0 < b$, then the barrier policy (L, U) is obtained by applying to X the two-sided regulator, so that

- (1) L and U are continuous and
- (2) L and U increase only when $Z = 0$ and $Z = b$, respectively.

If $X_0 > b$, we take $U_0 = X_0 - b$, and then future increments of (L, U) are determined by applying the two-sided regulator to $X - U_0$ in the obvious way. For the following proposition, let $\alpha_*(\lambda)$ and $\alpha^*(\lambda)$ be defined as in §3.2.

(3) Proposition. Let $g(x) \equiv \alpha_*(\lambda)e^{\alpha_*(\lambda)x} + \alpha^*(\lambda)e^{-\alpha^*(\lambda)x}$ for $x \in R$. The value function for the barrier policy (L, U) with parameter $b > 0$ is

$$(4) \quad v(x) = rg(x)/g'(b) + cg(x - b)/g'(-b), \quad 0 \leq x \leq b,$$

and

$$(5) \quad v(x) = v(b) + (x - b)r, \quad x > b.$$

Proof. Recall from §3.2 that the functions $f_1(x) \equiv \exp\{\alpha^*(\lambda)x\}$ and $f_2(x) \equiv \exp\{-\alpha_*(\lambda)x\}$ both satisfy $\lambda f - \Gamma f = 0$. Thus $\lambda g - \Gamma g = 0$, and obviously $g'(0) = 0$, so the function v defined by (4) satisfies

$$(6) \quad \lambda v(x) - \Gamma v(x) = 0, \quad 0 \leq x \leq b,$$

$$(7) \quad v'(0) = c \quad \text{and} \quad v'(b) = r.$$

It then follows from Corollary (5.2.4) that v is the desired value function on $[0, b]$. Finally, (5) is immediate from the definition of barrier policies. $\square$

For future reference, let us note that v' is continuous on $[0, \infty)$, whereas v'' may have a discontinuity at b .

§3. HEURISTIC DERIVATION OF THE OPTIMAL BARRIER

Assuming that there exists an optimal barrier policy, how should we set b ? The following is a heuristic argument to suggest the answer. Let (L, U) be the barrier policy corresponding to some given b value; call this the *nominal policy*. Let Z and v be the corresponding controlled process and value function, respectively. Also set

$$(1) \quad T(y) \equiv \inf \{t \geq 0 : Z_t = y\}, \quad 0 \leq y \leq b,$$

and

$$(2) \quad \phi(x, y) \equiv E_x[e^{-\lambda T(y)}], \quad 0 \leq x, y \leq b.$$

Suppose that our controller, following the nominal barrier policy, starts in state x , and let y be another state such that $0 < y < x < b$. The expected present value of total net reward over $[0, \infty)$ is, of course, $v(x)$, and we define

$$(3) \quad u(x, y) \equiv \text{expected present value, when starting in state } x \text{ and following the nominal policy, of net rewards earned over the period } [0, T(y)].$$

From the strong Markov property (5.1.8) of (L, U, Z) , one may argue as in §5.3 that $v(x) = u(x, y) + \phi(x, y)v(y)$, so we have

$$(4) \quad u(x, y) = v(x) - \phi(x, y)v(y).$$

Continuing to assume $0 < y < x < b$, let ϵ be a perturbation, either positive or negative, small enough that $0 < y + \epsilon$ and $x + \epsilon < b$. Let the starting state be $x + \epsilon$, and consider the *alternate strategy* where one follows a barrier policy with parameter $b + \epsilon$ up until the first time $T^*(y + \epsilon)$ at which level $y + \epsilon$ is hit, and then reverts to usage of the nominal policy with barrier height b ever afterward. Let

$$(5) \quad v^*(x + \epsilon) \equiv \text{expected present value, starting at level } x + \epsilon, \text{ of net rewards earned under the alternate strategy over } [0, \infty).$$

From the spatial homogeneity of Brownian motion we obtain

$$(6) \quad \phi(x, y) = E_{x+\epsilon}[e^{-\lambda T^*(y + \epsilon)}],$$

and similarly,

$$(7) \quad u(x, y) = \text{expected present value, starting at level } x + \epsilon \text{ and following the alternate strategy, of net rewards earned over the period } [0, T^*(y + \epsilon)].$$

Thus as a precise analog of (4), we have

$$(8) \quad v^*(x + \epsilon) = u(x, y) + \phi(x, y)v(y + \epsilon) \\ = v(x) + \phi(x, y)[v(y + \epsilon) - v(y)].$$

The last equality is obtained by substitution of (4). Subtracting $v(x + \epsilon)$ from (8), we see that the *improvement* effected by the alternate strategy is

$$(9) \quad v^*(x + \epsilon) - v(x + \epsilon) = \phi(x, y)[v(y + \epsilon) - v(y)] \\ - [v(x + \epsilon) - v(x)].$$

If the nominal policy is to be optimal, this expression must have a local minimum at $\epsilon = 0$, which obviously requires $0 = \phi(x, y)v'(y) - v'(x)$. We have derived this condition for $0 < y < x < b$, but then by continuity it must be that

$$(10) \quad v'(x) = \phi(x, y)v'(y) \quad \text{for } 0 \leq y < x \leq b.$$

In particular, taking $x = b$ and $y = 0$, one can use the boundary conditions (2.7) to deduce from (10) that

$$(11) \quad r = \phi(b, 0)c.$$

To repeat, (10) looks to be a necessary condition for optimality of the nominal policy, and (11) is a special case of (10). Of course, (11) uniquely determines b , but now we need to do a calculation. It was shown in Problem 5.4 that $\phi(x, b) = g(x)/g(b)$, $0 \leq x \leq b$, where g is defined as in (2.3). It can be shown in exactly the same way that

$$(12) \quad \phi(x, 0) = \frac{g(x - b)}{g(-b)}, \quad 0 \leq x \leq b.$$

Combining (11) and (12), we seek a value $b > 0$ such that $g(0)/g(-b) = r/c$. To prove the following, readers need only check that g is strictly decreasing on $(-\infty, 0]$ with $g(-\infty) = \infty$ (remember that $0 < r < c$ by assumption).

(13) Proposition. Let g be defined as in (2.3). There exists a unique $b > 0$ such that $g(0)/g(-b) = r/c$.

Preliminary to proving the optimality of this policy, we need the following.

(14) Proposition. Suppose b is chosen as in (13), and let $v(\cdot)$ be the value function for the corresponding barrier policy. Then $v'(x) = cg(x - b)/g(-b)$, $0 \leq x \leq b$.

(15) Corollary. Under the same hypotheses, v is concave, increasing, and twice continuously differentiable on $[0, \infty)$. Also $\lambda v(x) - \Gamma v(x) \geq 0$, $x \geq 0$.

Proof. To ease notation, set $f(x) \equiv cg(x - b)/g(-b)$, $0 \leq x \leq b$. Recall from the proof of (2.3) that $\lambda v - \Gamma v = 0$ on $[0, b]$. Differentiating both sides of this equation shows that $\lambda v' - \Gamma v' = 0$ on $[0, b]$ and (2.7) says that $v'(0) = c$ and $v'(b) = r$. Also recall from the proof of (2.3) that $\lambda g - \Gamma g = 0$ on R with $g'(0) = 0$. Here we have chosen b so that $g(0)/g(-b) = r/c$. Thus $\lambda f - \Gamma f = 0$ on $[0, b]$ with $f(0) = c$ and $f(b) = r$, implying that $v' = f$ on $[0, b]$. This proves (14).

For the corollary, first recall from (2.3) that v is linear with slope r on $[b, \infty)$. Readers may verify that g is decreasing on $(-\infty, 0]$ and we have already observed that $g'(0) = 0$. Thus f is decreasing on $[0, b]$ with $f(b) = r$ and $f'(b) = 0$. Because $v' = f$ on $[0, b]$, this means that v is concave on $[0, b]$ with $v'(b-) = r$ and $v''(b-) = 0$. But $v'(b+) = r$ and $v''(b+) = 0$ because of the linearity mentioned above, so v' and v'' are both continuous on $[0, \infty)$. For the last statement of the corollary, recall again that $\lambda v - \Gamma v = 0$ on $[0, b]$. Moreover, Γv is constant on $[b, \infty)$, whereas v is strictly increasing on $[b, \infty)$, which implies that $\lambda v(x) - \Gamma v(x) > 0$ for $x > b$. $\square$

§4. VERIFICATION OF OPTIMALITY

Now let (L, U) be an arbitrary feasible policy with $Z \equiv X + L - U$ and v the associated value function as in §2. Using the notational conventions of §4.7, we also set

$$(1) \quad A_t \equiv L_t - \sum_{0 < s \leq t} \Delta L_s \quad \text{for } t \geq 0$$

and

$$(2) \quad B_t \equiv U_t - \sum_{0 < s \leq t} \Delta U_s \quad \text{for } t \geq 0.$$

Thus A and B are continuous VF processes, called the continuous parts of L and U , respectively. Next let $f: [0, \infty) \rightarrow R$ be twice continuously differentiable with bounded derivative. We define $\Delta f(Z)_t \equiv f(Z_t) - f(Z_{t-})$ for $t > 0$ as in §4.7, and we extend this to $t = 0$ with the convention

$$(3) \quad \Delta f(Z)_0 \equiv f(Z_0) - f(X_0).$$

(4) Proposition. For any $T > 0$ and $x \geq 0$,

$$\begin{aligned} E_x[e^{-\lambda T} f(Z_T)] &= f(x) + E_x \left[\int_0^T e^{-\lambda t} (\Gamma f - \lambda f)(Z) dt \right] \\ &\quad + E_x \left[\int_0^T e^{-\lambda t} f'(Z) d(A - B) \right] \\ &\quad + E_x \left[\sum_{0 \leq t \leq T} e^{-\lambda t} \Delta f(Z)_t \right]. \end{aligned}$$

Proof. Let X be represented as $X_t = X_0 + \sigma W_t + \mu t$, where W is a standard Brownian motion. From the generalized Ito formula (4.7.4) one obtains

$$(5) \quad \begin{aligned} f(Z_t) &= f(Z_0) + \int_0^t f'(Z) \sigma dW + \int_0^t \Gamma f(Z) ds \\ &\quad + \int_0^t f'(Z) d(A - B) + \sum_{0 < s \leq t} \Delta f(Z)_s. \end{aligned}$$

Also, the specialized integration by parts formula (4.8.6) says that

$$(6) \quad e^{-\lambda T} f(Z_T) = f(Z_0) + \int_0^T e^{-\lambda t} df(Z) - \lambda \int_0^T e^{-\lambda t} f(Z) dt.$$

Now use (5) to calculate the differential $df(Z)$, substitute this into (6), and collect similar terms to get

$$\begin{aligned}
(7) \quad e^{-\lambda T} f(Z_T) &= f(Z_0) + \sigma \int_0^T e^{-\lambda t} f'(Z) dW \\
&+ \int_0^T e^{-\lambda t} f'(Z) d(A - B) \\
&+ \int_0^T e^{-\lambda t} (\Gamma f - \lambda f)(Z_t) dt \\
&+ \sum_{0 \leq t \leq T} e^{-\lambda t} \Delta f(Z)_t.
\end{aligned}$$

Next, (3) gives

$$(8) \quad f(Z_0) + \sum_{0 \leq t \leq T} e^{-\lambda t} \Delta f(Z)_t = f(X_0) + \sum_{0 \leq t \leq T} e^{-\lambda t} \Delta f(Z)_t.$$

Substitute (8) into (7) and take E_x of both sides, noting that the stochastic integral has zero expectation because its integrand is bounded (see §4.3). This gives the desired formula. $\square$

Recall that v currently denotes the value function for the arbitrary feasible policy (L, U) . Throughout the remainder of this section we use f to denote the value function for the barrier policy whose parameter b is chosen as in (3.13). It follows from (3.14) and (3.15) that

$$(9) \quad f \text{ is twice continuously differentiable on } [0, \infty),$$

$$(10) \quad r \leq f'(x) \leq c \quad \text{for all } x \geq 0,$$

and

$$(11) \quad \Gamma f(x) - \lambda f(x) \leq 0 \quad \text{for all } x \geq 0.$$

Our ultimate objective here is to prove $v(x) \leq f(x)$ for all $x \geq 0$, which will prove the optimality of the barrier policy constructed in §3. As an intermediate step, let

$$(12) \quad v_T(x) \equiv E_x \left\{ \int_0^T e^{-\lambda t} (r dU - c dL) + e^{-\lambda T} f(Z_T) \right\}$$

for $T > 0$ and $x \geq 0$. This is the value function for a *hybrid policy* that follows (L, U) up to time T , yielding a system content of Z_T at that point, and then enforces the barrier policy with value function f thereafter.

(13) Proposition. With the assumptions and definitions above,

$$\begin{aligned}
v_T(x) &= f(x) + E_x \left\{ \int_0^T e^{-\lambda t} (\Gamma f - \lambda f)(Z_t) dt \right\} + E_x \left\{ \int_0^T e^{-\lambda t} [f'(Z) - c] dA \right\} \\
&+ E_x \left\{ \int_0^T e^{-\lambda t} [r - f'(Z)] dB \right\} \\
&+ E_x \left\{ \sum_{0 \leq t \leq T} e^{-\lambda t} [\Delta f(Z)_t - c \Delta L_t + r \Delta U_t] \right\}.
\end{aligned}$$

(14) Remark. For future reference, we express the right side of (13) as $f(x) + E_x[I_1(T) + I_2(T) + I_3(T) + \Sigma(T)]$.

Proof. This follows directly from (4) and (12), using the identities $dL = dA + \Delta L$ and $dU = dB + \Delta U$ in (12). $\square$

(15) Corollary. $v_T(x) \leq f(x)$ for all $T > 0$ and $x \geq 0$, and thus $v(x) \leq f(x)$ as well.

Proof. Using the notational convention (14), it is clear that (11) implies $E_x[I_1(t)] \leq 0$, whereas (10) implies $E_x[I_2(T) + I_3(T)] \leq 0$. Furthermore, (10) implies $E_x[\Sigma(T)] \leq 0$ as follows. Suppose $\Delta L_t > 0$ and $\Delta U_t = 0$. Then $\Delta Z_t = \Delta L_t$ and we have

$$(16) \quad \Delta f(Z)_t - c \Delta L_t + r \Delta U_t = f(Z_t) - f(Z_t - \Delta L_t) - c \Delta L_t.$$

The right-hand side of (16) is negative because $f'(\cdot) \leq c$. Because $f'(\cdot) \geq r$, a similar inequality is obtained for times t with $\Delta U_t \geq 0$ and $\Delta L_t = 0$, and readers may easily verify that the same conclusion holds when $\Delta U_t > 0$ and $\Delta L_t > 0$. Combining this with (13) shows that $v_T(x) \leq f(x)$. Now let $T \rightarrow \infty$ in the definition (12) of $v_T(x)$. The first term on the right approaches $v(x)$ and

$$\liminf e^{-\lambda T} f(Z_T) \geq 0 \text{ as } T \rightarrow \infty$$

because f is bounded below. Thus letting $T \rightarrow \infty$ in the inequality $v_T(x) \leq f(x)$, we obtain $v(x) \leq f(x)$ as desired. $\square$

We have now proved that the barrier policy of §3 is optimal. The style of argument used here is often described as *policy improvement logic*. Beginning with a candidate optimal policy having known value function f , we examine the effect of inserting some other policy over an interval $[0, T]$ and reverting to use of the candidate policy thereafter. The question is whether the candidate policy can be improved by such a modification. If not, optimality of the candidate policy follows easily, as we have just seen.

§5. CASH MANAGEMENT

As an application, let us consider the so-called *stochastic cash management problem*. Here Z_t represents the content of a cash fund into which various types of income or revenue are automatically channeled, and out of which operating disbursements are made. In our formulation, the net of such routine deposits less routine disbursements is modeled by a (μ, σ) Brownian motion. That is, in the absence of managerial intervention, the content of the fund fluctuates as a (μ, σ) Brownian motion X . Let us suppose that wealth not held as cash is invested in securities (hereafter called *bonds*) that pay interest continuously at rate λ . Denote by S_t the dollar value of bonds held at time t . At any point, money can be transferred from the cash fund to buy bonds, but a transaction cost of β dollars must be paid for each dollar so transferred. That is, management gets only $1 - \beta$ dollars' worth of bonds in exchange for one dollar of cash. Similarly, bonds can be sold at any time to obtain additional cash, but management must give up $1 + \alpha$ dollars' worth of bonds to obtain one dollar of cash. Management is obliged to keep the content of the cash fund positive, and the firm's initial wealth (cash plus bonds) is large enough that we can safely ignore the possibility of ruin.

Let U_t denote the cumulative amount of cash used to buy bonds up to time t , each dollar of which buys only $1 - \beta$ dollars' worth of bonds. Similarly, let L_t denote the cumulative amount of cash generated by sale of bonds up to time t , each dollar of which requires liquidation of $1 + \alpha$ dollars' worth of bonds. The content of the cash fund is then $Z_t = X_t + L_t - U_t$ at time t , with $X_0 \equiv Z_0$ by convention. The dynamics of the process S_t are given by

$$(1) \quad dS_t = \lambda S_t dt + (1 - \beta) dU_t - (1 + \alpha) dL_t,$$

which means that

$$(2) \quad S_T = S_0 e^{\lambda T} + \int_0^T e^{\lambda(T-t)} [(1 - \beta) dU - (1 + \alpha) dL].$$

Let us suppose that management seeks to maximize the expected total wealth $E(Z_T + S_T)$ at some specified distant time T . This is, of course, equivalent to maximizing the expected present value

$$(3) \quad e^{-\lambda T} E(S_T + Z_T).$$

It is easy to show that $\exp(-\lambda T)E(Z_T)$ vanishes as $T \rightarrow \infty$ under an optimal policy. Thus substituting (2) into (3), sending $T \rightarrow \infty$, and ignoring the uncontrollable term S_0 , we arrive at the objective of *maximizing*

$$(4) \quad E \left\{ \int_0^\infty e^{-\lambda t} [(1 - \beta) dU - (1 + \alpha) dL] \right\}.$$

This is the equivalent maximization problem derived in §1 with $r = 1 - \beta$ and $c = 1 + \alpha$. Using integration by parts and the definition $Z \equiv X + L - U$, one can reverse the logic used in §1 to show that maximization of (4) is equivalent to *minimization* of

$$(5) \quad E \left\{ \int_0^\infty e^{-\lambda t} [\lambda Z_t dt + \alpha dL + \beta dU] \right\},$$

which is how our stochastic control problem was originally formulated in §2. Here the problem parameters α and β represent actual out-of-pocket transaction costs, whereas the holding cost parameter $h = \lambda$ reflects an *opportunity loss* on assets held as cash.

NOTES AND COMMENTS

This chapter is based on Harrison–Taylor (1978) and Harrison–Taksar (1983). The problem and its solution originally appeared in the former paper, whereas the methods used here are those of the latter paper. We have seen that the optimal controls (L, U) are continuous, but their points of increase form a set of (Lebesgue) measure zero. Control problems whose optimal processes have this property are sometimes called *singular*. Other singular control problems have been studied by Bather–Chernoff (1966), Benes–Shepp–Witsenhausen (1980), Karatzas (1981), and Shreve–Lehoczky–Gaver (1983).

Suppose that, in addition to the constraints and costs described in §1, each deposit entails a fixed cost of size K , and each withdrawal entails a fixed cost of size M . The optimal policy is then described by three critical numbers:

The controller makes a deposit of size q whenever level zero is hit, and reduces the storage level to Q whenever level S is hit ($0 < q < Q < S$). This statement is proved and the critical numbers are calculated in Harrison–Sellke–Taylor (1983). Problems of this type, where the optimal policy involves only the enforcement of jumps at isolated points in time, are called *impulse control problems*.

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CHAPTER 7

Optimizing Flow System Performance

To illustrate the use of regulated Brownian motion as a flow system model, let us return to the problem posed in §2.5. There we considered a single-product firm that must fix its work force size, or production capacity, at time zero. Having fixed its capacity, the firm may choose an actual production rate at or below this level in each future period, but overtime production is initially assumed impossible. Demand that cannot be met from stock on hand is lost with no adverse effect on future demand. For purposes of illustration, suppose that the number of units demanded in successive weeks are independent and identically distributed random variables with a mean and standard deviation of

$$a = 1000 \quad \text{and} \quad \sigma = 200,$$

respectively. As in §2.5, let π denote the selling price per unit of finished goods, w the labor cost parameter, and m the materials cost per unit. The firm pays w dollars each week for each unit of potential production (capacity), regardless of whether that potential is fully exploited, so the variable cost of production after time zero is m . Let us suppose that

$$\pi = \$130, \quad w = \$20, \quad \text{and} \quad m = \$50.$$

Also, as the interest rate for discounting, let

$$\lambda = 0.005 \text{ (one half of 1\% per week).}$$

With continuous compounding, the equivalent *annual* interest rate is $\exp(52\lambda) - 1 = 0.297$, or approximately 30%. Readers should note that

the values of a , σ , and λ all reflect our choice of the week as time unit. Finally, for the physical cost of holding inventory, let $p = \$0.10$ per week. These values give a contribution margin and effective holding cost of

$$\delta \equiv \pi - m = \$80 \quad \text{and} \quad h \equiv p + m\lambda = \$0.35 \text{ per week}.$$

If σ were zero, then the firm would set its production capacity precisely equal to the level demand rate of $a = 1000$ units per week, and would realize a weekly profit of $a(\pi - w - m) = \$60,000$. (If there are fixed costs of plant, equipment, and the like, then this is not really profit in the usual sense, but we shall ignore such considerations.)

As before, let B_t denote cumulative demand up to time t , and let μ denote the *excess capacity* (possibly negative) decided on at time zero. Thus cumulative potential production over $[0, t]$ is $A_t \equiv (a + \mu)t$. Throughout this chapter we assume that the centered demand process $\{B_t - at, t \geq 0\}$ can be adequately approximated by a $(0, \sigma)$ Brownian motion. With excess capacity μ , the netput process $X \equiv A - B$ is then modeled as a (μ, σ) Brownian motion. In §2.5 it was shown that maximizing the expected present value of total profit is equivalent to *minimizing*

$$(1) \quad \Delta \equiv E \left\{ \int_0^\infty e^{-\lambda t} [hZ \, dt + w\mu \, dt + \delta \, dL] \right\},$$

where $Z_t \equiv X_t + L_t - U_t$ is the inventory level at time t , L_t is the cumulative potential sales lost over $[0, t]$ due to stockouts, and U_t is the cumulative amount of undertime employed (potential production foregone) up to time t . The firm's capacity decision corresponds to choosing a μ value at time zero, and its dynamic operating policy is manifested in the choice of an undertime process U . These two aspects of management policy influence lost sales, of course, because L must increase fast enough to ensure $X_t + L_t - U_t \geq 0$ for all t .

Throughout this chapter we assume $Z_0 = 0$. Given a choice of μ , the dynamic production control problem can be formulated exactly as in Chapter 6. It might be argued that this formulation endows the firm with unrealistically broad control capabilities, such as the ability to effect instantaneous jumps in the inventory level. However, the development in Chapter 6 shows that such capabilities are never used, even if assumed available, because the optimal policy has a single-barrier form. That is, the optimal pair (L, U) enforces a lower control barrier at zero and an upper control barrier at b . In terms of the physical system, this means that potential production is foregone (capacity is underutilized) only as necessary to keep $Z \leq b$, and potential sales are lost when Z reaches zero.

Given the optimality of single-barrier policies, our problem amounts to choosing the parameters of a Brownian flow system (L, U, Z) . More specifically, we seek values for μ and b that minimize the performance measure Δ defined by (1). Using results from Chapter 6, one can write an explicit formula for Δ in terms of μ and b , then determine the optimal parameter values with a straightforward numerical search. This will be done in the next section. Before proceeding, readers may wish to make at least an order-of-magnitude guess as to the optimal policy. Should μ be positive or negative, and roughly what should be its magnitude as a percentage of the average demand rate? Under the optimal production control policy, roughly what fraction of demand will be lost? How big should the average inventory be, expressed as a multiple of average weekly demand? Roughly what is the cost of stochastic variability as a percentage of the weekly profit level achievable in the deterministic case?

§1. EXPECTED DISCOUNT COST

Suppose that μ and the inventory limit $b > 0$ are specified. Combining Propositions (6.1.8) and (6.2.3), we then have (remember $Z_0 = 0$ by assumption)

$$(1) \quad \Delta = \frac{h\mu}{\lambda^2} + \frac{w\mu}{\lambda} - \frac{rg(0)}{g'(b)} - \frac{cg(-b)}{g'(-b)},$$

where $r \equiv h/\lambda = 0.35/0.005 = \70 , $c \equiv h/\lambda + \delta = \$70 + \$80 = \150 , and

$$(2) \quad g(x) \equiv \alpha_*(\lambda)e^{\alpha^*(\lambda)x} + \alpha^*(\lambda)e^{-\alpha_*(\lambda)x}, \quad x \in R,$$

as in §6.2. Values of $\lambda\Delta$ are displayed in Table 1 for various choices of μ and b . Recall from §2.5 that Δ represents the *degradation of system performance*, in terms of expected present value, *from a deterministic ideal*. The performance measure $\lambda\Delta$ expresses this degradation in *equivalent annuity terms*. It is a constant rate of cost (here in thousands of dollars per week) that, if continued forever, is equivalent to a lump-sum cost of Δ at time zero. The cost figures in Table 1 should be compared against the ideal profit level of \$60,000 per week calculated earlier for the case where $\sigma = 0$. For all the parameter combinations considered in Table 1, we see a performance degradation between 2 and 5%.

The most attractive combination of parameter values appearing in Table 1 is $\mu = 10$ and $b = 2500$, yielding a performance degradation of \$1270 per

Table 1. Values of $\lambda\Delta$ (in thousands of dollars per week) with Original Data

	$b = 1500$	$b = 2000$	$b = 2500$	$b = 3000$	$b = 3500$
$\mu = -20$	2.03	1.90	1.85	1.83	1.82
$\mu = -10$	1.73	1.58	1.52	1.51	1.50
$\mu = 0$	1.51	1.37	1.33	1.33	1.34
$\mu = 10$	1.39	1.28	1.27	1.31	1.35
$\mu = 20$	1.35	1.30	1.33	1.40	1.49
$\mu = 30$	1.39	1.39	1.47	1.57	1.68
$\mu = 40$	1.48	1.53	1.64	1.76	1.88
$\mu = 50$	1.61	1.70	1.83	1.97	2.10
$\mu = 60$	1.77	1.89	2.03	2.17	2.31

week (to three significant figures). A finer-scale search shows that only trivial improvements on this performance are possible, and hence the pair ($\mu = 10$, $b = 2500$) will hereafter be called optimal. An excess capacity of 10 units per week amounts to 1% of the average demand rate and thus the optimal system configuration calls for a high degree of balance between production and demand. It will be shown in §4 that the long-run average inventory is about 1500 units, or 1.5 weeks of average demand. In different terms, the average inventory is about 7.5 times the standard deviation of weekly demand. In §4 it will be shown that this high level of buffer stock results in a long-run average lost demand rate below 1%.

For each value of μ , the optimal barrier b can be calculated as in §6.3, but this is really no more efficient than a direct numerical search using the performance measure $\lambda\Delta$. If we assumed a positive initial inventory, the optimal value of b for each fixed μ would be unchanged (see Chapter 6), but the optimal value of μ would generally be lower. Finally, Table 1 shows that system performance is quite insensitive to changes in b and μ , which suggests that nearly optimal performance might be obtained with a much cruder mode of analysis. That idea will be pursued further in §5.

§2. OVERTIME PRODUCTION

In the spirit of Problem 2.8, let us now suppose that overtime production is possible in essentially unlimited quantities, and that such production is essentially instantaneous. That is, whenever inventory falls to zero, management can order overtime production fast enough and in large enough quantities to completely avoid lost demand. The penalty is that a premium wage rate

Table 2. Values of $\lambda\Delta$ (in thousands of dollars per week) with Overtime Production Capability

	$b = 1000$	$b = 2000$	$b = 3000$	$b = 4000$	$b = 5000$
$\mu = -60$	0.83	0.76	0.76	0.76	0.76
$\mu = -50$	0.78	0.70	0.69	0.69	0.69
$\mu = -40$	0.75	0.64	0.64	0.64	0.64
$\mu = -30$	0.73	0.61	0.60	0.60	0.60
$\mu = -20$	0.74	0.60	0.60	0.61	0.61
$\mu = -10$	0.77	0.64	0.66	0.68	0.70
$\mu = 0$	0.82	0.72	0.78	0.84	0.90
$\mu = 10$	0.90	0.84	0.96	1.08	1.19
$\mu = 20$	1.00	1.00	1.18	1.36	1.51

$$w^* = 1.5 w = \$30$$

must be paid for each unit of overtime production. The structure of our two-stage decision problem is unchanged, but now L_t must be interpreted as the cumulative overtime production up to time t . The degradation of system performance from its deterministic ideal is again given by formula (0.1), and hence (1.1), except that the lost contribution δ is replaced by the overtime wage rate w^* . (Readers were asked to verify this statement in Problem 2.8.) Values of $\lambda\Delta$ for different combinations of μ and b are shown in Table 2.

According to Table 2, the minimal performance degradation is about \$600 per week (1% of the deterministic ideal profit level), achievable with a variety of different parameter combinations. A finer-scale search shows that only trivial improvements on this performance are possible, so the pair ($\mu = -20$, $b = 3000$) will hereafter be called optimal. The $\lambda\Delta$ value of 600 represents an improvement of $\$1270 - \$600 = \$670$ per week over the base case treated in §1.

With the ability to schedule modest amounts of overtime production, the optimal regular-time capacity is about 2% below average demand, compared with a regular-time capacity about 1% higher than average demand in the base case of §1. Thus total employment decreases by about 3%, but the average wage rate increases slightly due to overtime premiums. See §4 for further calculations of this type.

§3. HIGHER HOLDING COSTS

With our original data, the financial cost of holding inventory is $\lambda m = \$0.25$ per unit per week as compared with a physical holding cost of $p = \$0.10$.

Table 3. Values of $\lambda\Delta$ (in thousands of dollars per week) with Higher Holding Costs

	$b = 500$	$b = 1000$	$b = 1500$	$b = 2000$	$b = 2500$
$\mu = -30$	4.29	3.08	2.84	2.79	2.79
$\mu = -20$	4.02	2.77	2.52	2.48	2.49
$\mu = -10$	3.78	2.53	2.29	2.28	2.33
$\mu = 0$	3.58	2.34	2.15	2.20	2.32
$\mu = 10$	3.41	2.23	2.11	2.25	2.47
$\mu = 20$	3.27	2.17	2.15	2.39	2.71
$\mu = 30$	3.16	2.17	2.25	2.59	2.99
$\mu = 40$	3.09	2.22	2.41	2.82	3.28
$\mu = 50$	3.04	2.31	2.59	3.06	3.56

Returning to the assumption that overtime production is impossible, let us now suppose $p = \$1.00$, which might correspond to a situation where inventory must be refrigerated or closely guarded. Table 3 gives values for $\lambda\Delta$ with these new data ($h = 1.25$). The most attractive parameter combination in the table is $\mu = 10$ and $b = 1500$, yielding a performance degradation of \$2110 per week. A finer-scale search shows that this level of performance cannot be significantly improved, so the pair ($\mu = 10$, $b = 1500$) will hereafter be called optimal.

With higher holding costs, the firm chooses roughly the same capacity level (work force size) that proved optimal in the base case of §1, but inventory is controlled more tightly, and thus more demand is ultimately lost (see §4). System performance worsens by $\$2100 - \$1270 = \$830$ per week.

§4. STEADY-STATE CHARACTERISTICS

Consider an arbitrary pair of values for μ and b . In §5.4 and 5.5 it was shown that $E(L_t)/t \rightarrow \alpha$, $E(U_t)/t \rightarrow \beta$, and $E(Z_t) \rightarrow \gamma$ as $t \rightarrow \infty$, where

$$(1) \quad \alpha = \frac{\sigma^2}{2b} \psi\left(\frac{2\mu b}{\sigma^2}\right) \quad \text{and} \quad \beta = \mu + \alpha,$$

$$(2) \quad \gamma = b\phi\left(\frac{2\mu b}{\sigma^2}\right),$$

$$(3) \quad \psi(\xi) \equiv \frac{\xi}{e^\xi - 1},$$

and

$$(4) \quad \phi(\xi) \equiv \frac{e^{-\xi} - 1 + \xi}{\xi(1 - e^{-\xi})}.$$

For the three combinations of μ and b that proved optimal in §1 to 3, the values of α , β , and γ are displayed in Table 4. In all three cases, γ represents the long-run average inventory level, and β is the average amount of regular-time capacity that goes unused each week. In both §1 and §3, L represented cumulative lost demand, so one interprets α as the average rate at which demand is lost (in units per week). But L represented cumulative overtime production in §2, so there α is interpreted as the long-run average rate of overtime production (again in units per week).

With respect to the original problem treated in §1, Table 4 shows that only 0.4% of demand is lost, that about 1.4% of the labor paid for is not used, and that average inventory is about 1500. With the higher holding costs assumed in §3, management chooses the same capacity level (work force size) but controls inventory more tightly. This reduces average inventory to 843 units, and still less than 1% of demand is lost.

For the problem treated in §2, regular-time capacity is set 2% below the average demand rate, only 0.1% of this regular-time capacity goes unused, and then 2.1% of demand is satisfied with overtime production. By setting the regular-time production rate below average demand, a relatively low average inventory level is achieved despite the loose inventory limit of $b = 3000$. System performance would not be very much different, in fact, if we took $b = \infty$.

Note that the firm's weekly wage payments with our original assumptions amount to $1010w$ dollars, and the corresponding figure under the assumptions of §2 is $980w + 21w^* = 980w + 21(1.5)w \approx 1010w$. Thus the firm pays about the same for labor when overtime is available at the usual 50% wage premium, but the ability to get additional production just when it is needed leads to an inventory reduction that substantially improves overall performance.

Table 4. Steady-State Characteristics of Optimal Policies for Various Cases

Case	μ	b	α	β	γ
Original data (§1)	10	2500	4.01	14.01	1503
Overtime capability (§2)	-20	3000	21.0	1.0	633
Higher holding cost (§3)	10	1500	8.95	18.95	843

§5. AVERAGE COST CRITERION

As stated earlier in §2.5, our discounted performance measure $\lambda\Delta$ can be approximated by the long-run average cost rate

$$(1) \quad \rho \equiv \delta\alpha + w\mu + h\gamma$$

when λ is small. (Readers should note that λ still figures in the computation of ρ through the relation $h \equiv p + m\lambda$.) In operations research it is usual, or at least common, to take as primitive the objective of minimizing ρ . Substituting formulas (4.1) and (4.2) into (1) gives

$$(2) \quad \rho = \frac{\delta\sigma^2}{2b} \psi\left(\frac{2\mu b}{\sigma^2}\right) + w\mu + hb\phi\left(\frac{2\mu b}{\sigma^2}\right).$$

Under the original assumptions of §1, we have $\delta = \$80$, $w = \$20$, and $h = \$0.35$. Values of ρ , calculated using these cost data and formula (2), are displayed in Table 5. The most attractive parameter combination in the table is $\mu = 10$ and $b = 2500$, which is the same pair found optimal in §1.

For the case treated in §2, we calculate ρ exactly as above, except that $w^* \equiv 30$ now plays the role of δ . In a search of the same (μ, b) pairs considered in Table 2, the minimal ρ value is achieved at $(\mu = -20, b = 3000)$, which was also best under the original discounted criterion. Finally, using the cost assumptions of §3, a search of the same (μ, b) pairs considered in Table 3 shows that ρ is minimal for the same pair $(\mu = 10, b = 1500)$ that was found optimal earlier. In each of our three cases, if one searches on a finer scale under both the discounted and average cost criteria, the two

Table 5. Values of ρ (in thousands of dollars per week) with Original Data

	$b = 1500$	$b = 2000$	$b = 2500$	$b = 3000$	$b = 3500$
$\mu = -20$	1.86	1.69	1.61	1.58	1.56
$\mu = -10$	1.55	1.36	1.27	1.23	1.21
$\mu = 0$	1.33	1.15	1.07	1.05	1.14
$\mu = 10$	1.21	1.07	1.05	1.08	1.15
$\mu = 20$	1.19	1.11	1.15	1.24	1.36
$\mu = 30$	1.24	1.23	1.32	1.46	1.61
$\mu = 40$	1.35	1.40	1.53	1.69	1.85
$\mu = 50$	1.49	1.59	1.74	1.91	2.09
$\mu = 60$	1.67	1.80	1.96	2.13	2.31

optima do not coincide exactly, but the (μ, b) pair that minimizes ρ is found to achieve a $\lambda\Delta$ value within 1% of the minimum. Thus minimizing ρ is effectively equivalent to minimizing $\lambda\Delta$, at least with the data used here.

Table 5 suggests that ρ is quite insensitive to our choice of μ and b , and thus it is natural to look for rough-and-ready formulas that will give approximately optimal parameter values. In that spirit, simply assume that the optimal μ value is zero. Because $\psi(0) = 1$ and $\phi(0) = 1/2$, equation (2) then reduces to $\rho = \delta\sigma^2/2b + hb/2$. Differentiating this with respect to b and setting the derivative equal to zero gives

$$(3) \quad b^* = \sigma\sqrt{\delta/h}.$$

Now consider optimization with respect to μ . Tedious calculation from (4.3) and (4.4) shows that $\psi'(0) = -1/2$, $\psi''(0) = 1/6$, $\phi'(0) = 1/12$, and $\phi''(0) = 0$. Thus for small values of ξ one has

$$(4) \quad \psi(\xi) = 1 - 1/2 \xi + 1/6 \xi^2 + o(\xi^2),$$

$$(5) \quad \phi(\xi) = 1/2 + 1/12 \xi + o(\xi^2).$$

Substituting (4) and (5) into (2) gives

$$(6) \quad \rho = \left(\frac{\delta\sigma^2}{2b} + \frac{hb}{2}\right) + \left(w - \frac{\delta}{2} + \frac{hb^2}{6\sigma^2}\right)\mu + \left(\frac{\delta b}{3\sigma^2}\right)\mu^2 + o(\mu^2)$$

for small μ and fixed b . Ignoring the $o(\mu^2)$ term in (6), we differentiate with respect to μ and set the derivative to zero to get

$$(7) \quad \mu = \frac{3\sigma^2}{2\delta b} \left(\frac{\delta}{2} - w - \frac{hb^2}{6\sigma^2}\right).$$

Finally, set $b = b^*$ in (7) and use (3) to arrive at

$$(8) \quad \mu^* = \frac{\sigma}{2} \left(\frac{h}{\delta}\right)^{1/2} \left(1 - \frac{3w}{\delta}\right).$$

Table 6 shows that the policy (μ^*, b^*) performs remarkably well for each of the three cases considered earlier. In each case, μ^* and b^* are calculated from (3) and (8), respectively, then $\lambda\Delta$ is calculated using (1.1) with param-

Table 6. Performance of Policy (μ^* , b^*) for Various Cases

Case	μ^*	b^*	$\lambda\Delta$	Error
Original data (§1)	1.66	3024	1.32	4%
Overtime capability (§2)	-10.8	1851	0.66	10%
Higher holding cost (§3)	3.13	1600	2.15	5%

ters μ^* and b^* . The error reported in Table 6 is the difference between this $\lambda\Delta$ value and the corresponding minimal value calculated earlier, expressed as a percentage of the minimal value. Readers wishing to check these calculations should remember that $w^* \equiv 30$ plays the role of δ in the problem treated in §2, and $h = 1.25$ in the problem of §3.

APPENDIX A

Stochastic Processes

The first three sections of this appendix are concerned with notation and terminology. Readers should particularly note the standing assumptions such as joint measurability of stochastic processes. The last two sections are brief, stating without proof a basic result from martingale theory and a useful version of Fubini's theorem.

§1. A FILTERED PROBABILITY SPACE

In the mathematical theory of probability, one begins with an abstract space Ω , a σ -algebra $\mathcal{F}$ on Ω , and a σ -additive probability measure P on $(\Omega, \mathcal{F})$. The pair $(\Omega, \mathcal{F})$ is called a *measurable space* and the triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*. Individual points $\omega \in \Omega$ represent possible *outcomes* for some experiment (broadly defined) in which we are interested. Identifying an appropriate outcome space Ω is always the first step in probabilistic modeling. Then $\mathcal{F}$ specifies the set of all *events* (subsets of Ω) to which we are prepared to assign probability numbers. Finally, the probability numbers $P(\cdot)$ reflect the relative likelihood of various events, whatever that may be interpreted to mean, and their specification is the second major step in probabilistic modeling. In economics one frequently interprets the probability measure P as a quantification of the subjective uncertainty experienced by some rational economic agent. Most physical scientists feel the need for a stronger, objective interpretation related to physical frequency. See Savage (1954) or de Finetti (1974) for a systematic development of the subjective view of probability, and Fine (1973) for a survey of alternative objective views.

In this book, we usually take as primitive a probability space $(\Omega, \mathcal{F}, P)$ and a family $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ of σ -algebras on Ω such that (a) $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \geq 0$ and (b) $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. It is usual to express (a) and (b) by saying that $\mathbb{F}$ is an increasing family of sub- σ -algebras, or a *filtration* of $(\Omega, \mathcal{F})$. As a model element, $\mathbb{F}$ shows how information arrives (how uncertainty is resolved) as time passes. One interprets $\mathcal{F}_t$ as the set of all events whose occurrence or nonoccurrence will be determinable at time t . Let $\mathcal{F}_\infty$ denote the smallest σ -algebra on Ω containing all events in $\mathcal{F}_t$ for all $t \geq 0$. Without significant loss of generality, we shall assume that the ambient σ -algebra is $\mathcal{F} = \mathcal{F}_\infty$. Whenever we describe $(\Omega, \mathcal{F}, P)$ as a *filtered probability space*, it is understood that $(\Omega, \mathcal{F}, P)$ is a probability space, that $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is a filtration of $(\Omega, \mathcal{F})$, and that $\mathcal{F} = \mathcal{F}_\infty$.

Let $(\Omega, \mathbb{F}, P)$ be a filtered probability space. A *stopping time* is a measurable function T from $(\Omega, \mathbb{F})$ to $[0, \infty]$ such that $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. Note that this definition involves the filtration in a fundamental way; one should really say that T is a stopping time *with respect to* $\mathbb{F}$. It is often useful to think of T as a plan of action; our definition requires that the decision to stop at or before time t depend only on information available at t . Now let $\mathcal{F}_T$ consist of all events $A \in \mathcal{F}$ such that

$$(1) \quad \{\omega \in \Omega : \omega \in A \text{ and } T(\omega) \leq t\} \in \mathcal{F}_t$$

for all $t \geq 0$. Condition (1) is more compactly expressed by saying that $A \cap \{T \leq t\} \in \mathcal{F}_t$, and this level of symbolism will be used hereafter. One may think of $\mathcal{F}_T$ as the set of all events whose occurrence or nonoccurrence is known at the time of stopping.

§2. RANDOM VARIABLES AND STOCHASTIC PROCESSES

Recall that R denotes the real line and $\mathcal{B}$ is the Borel σ -algebra on R (the smallest σ -algebra containing all the open sets). Given a probability space $(\Omega, \mathcal{F}, P)$, a *random variable* is a measurable function X from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B})$. Thus to each outcome $\omega \in \Omega$ there corresponds a numerical value $X(\omega)$, which we call the *realization* of X for outcome ω . One may, of course, identify or define many different random variables on a single outcome space, this identification reflecting different aspects of the experimental outcome that are of interest to the model builder. The *distribution* of X is defined as the probability measure Q on $(R, \mathcal{B})$ given by

$$(1) \quad Q(A) \equiv P\{X^{-1}(A)\} \equiv P\{\omega \in \Omega : X(\omega) \in A\}, \quad A \in \mathcal{B}.$$

The corresponding *distribution function* F is given by

$$F(x) \equiv P\{\omega \in \Omega : X(\omega) \leq x\} \equiv Q(-\infty, x], \quad x \in R.$$

It is well known that F uniquely determines Q . Because the notion of function is more elementary than that of measure, it is usual to speak in terms of F rather than Q , but it will be seen shortly that the definition of the latter generalizes more readily.

It is customary to define a (one-dimensional) stochastic process as a family of random variables $X = \{X_t, t \in T\}$, where T is an arbitrary index set. (Elements of T usually represent different points in time.) For our purposes, the preceding definition will be specialized in two ways. First, the index set (or time domain) will always be $T = [0, \infty)$. Second, attention will be restricted to processes X that are *jointly measurable*. The meaning of this is as follows. Let $\mathcal{B}[0, \infty)$ denote the Borel σ -algebra on $[0, \infty)$ and let λ denote Lebesgue measure. Starting with a probability space $(\Omega, \mathcal{F}, P)$, recall the definitions of the product σ -algebra $\mathcal{F} \times \mathcal{B}[0, \infty)$ and product measure $P \times \lambda$. For our purposes, a stochastic process, or just *process*, is a mapping $X : \Omega \times [0, \infty) \rightarrow R$ that is measurable with respect to $\mathcal{F} \times \mathcal{B}[0, \infty)$. To denote the value of X at a point $(\omega, t) \in \Omega \times [0, \infty)$ we write either $X(\omega, t)$ or $X_t(\omega)$ with a consistent preference for the latter notation. It is a standard result in measure theory (usually stated as part of Fubini's theorem) that $X(\omega) \equiv \{X_t(\omega), t \geq 0\}$ is a Borel measurable function $[0, \infty) \rightarrow R$ for each fixed ω . Similarly, $X_t : \Omega \rightarrow R$ is an $\mathcal{F}$ -measurable function (a random variable) for each fixed t . The function $X(\omega)$ is called the *realization*, or *trajectory*, or *sample path* of the process X corresponding to outcome ω .

Our next topic is continuous processes for which some preliminary definitions are necessary. Let $C \equiv C[0, \infty)$ be the space of all continuous functions $x : [0, \infty) \rightarrow R$. (Functions are here denoted by letters like x and y , rather than the usual f and g , because we are thinking of them as sample paths of stochastic processes.) The standard metric ρ on this space is defined by

$$(2) \quad \rho_t(x, y) \equiv \sup_{0 \leq s \leq t} |x(s) - y(s)|, \quad t \geq 0,$$

$$(3) \quad \rho(x, y) \equiv \sum_{n=1}^{\infty} \frac{(1/2)^n \rho_n(x, y)}{1 + \rho_n(x, y)}$$

for $x, y \in C$. Note that ρ_t is the usual metric of uniform convergence on $C[0, t]$. When we say that $x_n \rightarrow x$ in C , this means that $\rho(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. The following is immediate from (2) and (3).

(4) Proposition. $x_n \rightarrow x$ in C if and only if $\rho_t(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $t > 0$.

One may paraphrase (4) by saying that ρ induces on C the topology of uniform convergence on finite intervals. A subset A of C is said to be *open* if for every point $x \in A$ there exists a radius $r > 0$ such that all $y \in C$ with $\rho(x, y) < r$ belong to A . As a precise analog of $\mathcal{B}$, we define $\mathcal{C}$ as the smallest σ -algebra on C containing all the open sets, calling $\mathcal{C}$ the Borel σ -algebra on C . See page 11 of Billingsley (1968) for some interesting commentary on this definition.

A stochastic process X on $(\Omega, \mathcal{F}, P)$ is said to be *continuous* if $X(\omega) \in C$ for all $\omega \in \Omega$. From this and the fact that X_t is measurable with respect to $\mathcal{F}$ for each $t \geq 0$, it can be shown that X is a measurable mapping $(\Omega, \mathcal{F}) \rightarrow (C, \mathcal{C})$. To put this in the language of Billingsley (1968), a continuous process may be viewed as a *random element* of the metric space C . The *distribution* of a continuous process X is the probability measure Q on $(C, \mathcal{C})$ defined by (1) with $\mathcal{C}$ in place of $\mathcal{B}$. One may paraphrase this definition by calling Q the probability measure on $(C, \mathcal{C})$ *induced from P by X* . It may be desirable to elaborate on this critically important concept. It can be verified that the sets

$$A \equiv \{x \in C : x(T) \leq a\}$$

and

$$B \equiv \{x \in C : x(t) \leq b, 0 \leq t \leq T\}$$

are both elements of $\mathcal{C}$. (Here a, b and $T > 0$ are constants.) Applying the definition of Q , we have

$$Q(A) = P\{\omega \in \Omega : X_T(\omega) \leq a\}$$

and

$$Q(B) = P\{\omega \in \Omega : M_T(\omega) \leq b\}$$

where $M_T(\omega) \equiv \sup\{X_t(\omega), 0 \leq t \leq T\}$. Suppressing the dependence on ω , as is usual in probability theory, these relations can be written as

$$Q(A) = P\{X_T \leq a\} \quad \text{and} \quad Q(B) = P\{M_T \leq b\}.$$

Thus knowledge of the process distribution Q gives us, at least in principle, not only the distributions of the individual random variables X_T but also

those of more complex functionals like maxima. It is an important fact that the process distribution Q is uniquely determined by the *finite-dimensional distributions* of X . This result will not be used here but interested readers are referred to Chapter 2 of Billingsley (1968) for further discussion.

Throughout this section, we have spoken of a stochastic process X defined on some probability space $(\Omega, \mathcal{F}, P)$. Enriching this setting, suppose that the probability space is equipped with a filtration $\mathbb{F}$. We say that X is an *adapted* process on the filtered probability space $(\Omega, \mathbb{F}, P)$ if X_t is measurable with respect to $\mathcal{F}_t$ for all $t \geq 0$. Heuristically, this means that the information available at time t includes the history of X up to that point.

§3. A CANONICAL EXAMPLE

To give a concrete example of a filtered probability space, and a stochastic process on that space, we take $\Omega = C$ and define the *projection maps* $X_t: C \rightarrow R$ via

$$X_t(\omega) = \omega(t) \quad \text{for } t \geq 0 \text{ and } \omega \in C.$$

(Here the generic element of C is denoted by a lower case Greek ω rather than the lower case Roman x used earlier for obvious reasons.) Now let $\mathcal{F}_t$ be the smallest σ -algebra on C with respect to which all the projections $\{X_s, 0 \leq s \leq t\}$ are measurable. Defining $\mathcal{F}_\infty$ in terms of $\{\mathcal{F}_t\}$ as in §1, it follows that $\mathcal{F}_\infty$ is the smallest σ -algebra with respect to which all the projections $\{X_t, t \geq 0\}$ are measurable. Moreover, it is shown on page 20 of Williams (1979) that $\mathcal{F}_\infty$ coincides with the Borel field $\mathcal{C}$ of §2. Now let P be any probability measure on $(C, \mathcal{C})$ and define $\mathbb{F} \equiv \{\mathcal{F}_t, t \geq 0\}$. Then $(\Omega, \mathbb{F}, P)$ is a filtered probability space in the sense of §1, and X is an adapted, continuous process on that space. Hereafter, we shall describe this canonical setup by saying that

- (1) Ω is path space,
- (2) X is the coordinate process on Ω , and
- (3) $\mathbb{F}$ is the filtration generated by X .

This canonical setup is appropriate when (a) the sample path of X is the only relevant source of uncertainty for current purposes and (b) the only relevant information available at time t is the history of X up to that point. (In general, when we say that a process X is adapted to a filtration $\mathbb{F}$, the

σ -algebra $\mathcal{F}_t$ may contain much more information than just the history of X up to time t .) Note that the coordinate process X , viewed as a mapping $C \rightarrow C$, is the identity map $X(\omega) = \omega$.

§4. MARTINGALE STOPPING THEOREM

Let X be a stochastic process on some filtered probability space $(\Omega, \mathcal{F}, P)$. This process X is said to be a *martingale* if it is adapted, $E(|X_t|) < \infty$ for all $t \geq 0$ and $E(X_t | \mathcal{F}_s) = X_s$ whenever $s \leq t$. This is yet another definition that involves the filtration in a fundamental way. A rich theory of martingales has developed in recent decades, but we shall need only the following modest result. It is a very special case of Doob's optional sampling theorem, which can be found in Liptser–Shiryaev (1978) and other recent books dealing with stochastic processes in continuous time.

(1) Martingale Stopping Theorem. Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space, T a stopping time on this space, and X a martingale with right-continuous sample paths. Then the stopped process $\{X(t \wedge T), t \geq 0\}$ is also a martingale.

From this it is immediate that $E[X(t \wedge T)] = E[X(0)]$ for any $t > 0$. If $P\{T < \infty\} = 1$ (hereafter written simply $T < \infty$), then, of course, $(t \wedge T) \rightarrow T$ almost surely as $t \rightarrow \infty$. We would like to conclude that $E[X(t \wedge T)] \rightarrow E[X(T)]$ as $t \rightarrow \infty$, and hence $E[X(T)] = E[X(0)]$. Unfortunately, this is not always true. For example, let X be a standard Brownian motion (zero drift and unit variance) with $X(0) = 0$, and let T be the first time t at which $X(t) = 1$. It is well known that X is a martingale (using the filtration generated by X itself), that T is a stopping time, and that $T < \infty$, but it is obviously false that $E[X(T)] = E[X(0)]$. The following proposition gives an easy sufficient condition for the desired conclusion.

(2) Corollary. In addition to the hypotheses of (1), suppose that $T < \infty$ and the stopped process $\{X(t \wedge T), t \geq 0\}$ is uniformly bounded. Then $E[X(T)] = E[X(0)]$.

Proof. Because $T < \infty$ almost surely, $X(t \wedge T) \rightarrow X(T)$ almost surely as $t \rightarrow \infty$. Because $\{X(t \wedge T), t \geq 0\}$ is bounded by hypothesis, the bounded convergence theorem shows that $E[X(t \wedge T)] \rightarrow E[X(T)]$ as $t \rightarrow \infty$. But from (1) we have $E[X(t \wedge T)] = E[X(0)]$ for all $t \geq 0$ and hence $E[X(T)] = E[X(0)]$. $\square$

§5. A VERSION OF FUBINI'S THEOREM

Recall the general statement of Fubini's theorem presented in Halmos (1974) and other basic texts on measure theory. Let $\mathcal{B}[0, \infty)$ and λ be as in §2, and let X be a stochastic process on $(\Omega, \mathcal{F}, P)$. Specializing Fubini's theorem to the product space $\Omega \times [0, \infty)$, the product measure $P \times \lambda$, and the jointly measurable function $X : \Omega \times [0, \infty) \rightarrow R$, we get the following theorem.

(1) Theorem. If $E[\int_0^\infty |X(t)| dt] < \infty$, then

$$E\left[\int_0^\infty X(t) dt\right] = \int_0^\infty E[X(t)] dt.$$

A closely related result, sometimes called *Tonelli's theorem*, says that (2) holds for positive processes X without any further hypotheses; in particular, the iterated integrals on the two sides are either both infinite or else both finite and equal.

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Real Analysis

This appendix collects several results from real analysis that play a central role in the text. Our standard references are Bartle (1976) and Royden (1968).

§1. ABSOLUTELY CONTINUOUS FUNCTIONS

Let $f: [0, \infty) \rightarrow R$ be fixed. The function f is said to be absolutely continuous on $[0, t]$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^n |f(b_i) - f(a_i)| < \epsilon$$

for every finite collection of nonoverlapping intervals $\{(a_i, b_i); i = 1, \dots, n\}$ with $0 \leq a_i < b_i \leq t$ and

$$\sum_{i=1}^n (b_i - a_i) < \delta.$$

When we say that f is absolutely continuous, this means that it is absolutely continuous on $[0, t]$ for every $t > 0$. The following is proved on page 106 of Royden (1968).

(1) Proposition. f is absolutely continuous if and only if there is a measurable function $g: [0, \infty) \rightarrow R$ such that $f(t) = f(0) + \int_0^t g(s) ds$ (Lebesgue integral), $t \geq 0$.

The function g appearing in (1) is called a *density* for f ; it is not unique, but any two densities must be equal except on a set of Lebesgue measure zero. Royden also shows on page 107 that an absolutely continuous function is differentiable almost everywhere (Lebesgue measure) and the derivative is a density.

§2. VF FUNCTIONS

Again let $f: [0, \infty) \rightarrow R$ be fixed. The total variation of f over $[0, t]$ is defined as

$$v_t(f) \equiv \sup \left\{ \sum_{i=1}^n |f(t_i) - f(t_{i-1})| \right\},$$

where the supremum is taken over all finite partitions $0 = t_0 < \dots < t_n = t$. We call f a VF function if $v_t(f) < \infty$ for all $t > 0$. (The acronym VF comes from the French literature on stochastic processes where it stands for *variation finite*.) The following important result can be found on page 100 of Royden (1968).

(1) Proposition. g is a VF function on $[0, \infty)$ if and only if it can be written as the difference of two increasing functions on $[0, \infty)$.

§3. RIEMANN-STIELTJES INTEGRATION

Starting with two functions $f, g: [0, \infty) \rightarrow R$, recall from Section 29 of Bartle (1976) what it means for f to be integrable with respect to g over $[0, t]$. This is analogous to the more familiar definition of Riemann integrability; $\int f dg$ will not exist unless f and g have quite a lot of structure. The following results can be found in Sections 29 and 30, respectively of Bartle.

(1) Integration by Parts Theorem. f is integrable with respect to g over $[0, t]$ if and only if g is integrable with respect to f over $[0, t]$. In this case,

$$(2) \quad \int_0^t f dg = [f(t)g(t) - f(0)g(0)] - \int_0^t g df.$$

(3) Integrability Theorem. If f is continuous and g is increasing over $[0, t]$, then f is integrable with respect to g over $[0, t]$.

In (2) and hereafter, we write $\int_0^t$ to signify a Riemann–Stieltjes integral over $[0, t]$. In the integral on the left side of (2), we call f the *integrand* and g the *integrator*. The indefinite integral $\int f dg$ will be understood to signify a function $h: [0, \infty) \rightarrow R$ defined by

$$h(t) \equiv \int_0^t f dg \quad \text{for } t \geq 0.$$

When we say that $\int f dg$ exists, this means that f is integrable with respect to g over every finite interval $[0, t]$. By combining (1), (2), and (2.1), we arrive at the following important result.

(4) Proposition. If f is continuous and g is a VF function, then $\int f dg$ and $\int g df$ both exist, and the integration by parts formula (2) is valid for all $t \geq 0$.

§4. THE RIEMANN–STIELTJES CHAIN RULE

The following result does not appear in Bartle, but one can easily construct a proof by generalizing that of Bartle's theorem 30.13.

(1) Chain Rule. Suppose that $f, g: [0, \infty) \rightarrow R$ are continuous, that g is a VF function, and that $\phi: R \rightarrow R$ is continuously differentiable. Then

$$(2) \quad \int_0^t f d\phi(g) = \int_0^t f\phi'(g) dg, \quad t \geq 0.$$

In formula (2), we write $\phi(g)$ to denote the function that has value $\phi(g(t))$ at time t . Similarly, $f\phi'(g)$ denotes the function that has value $f(t)\phi'(g(t))$ at time t ; the right side of (2) is the integral of this function with respect to g . It is immediate from (3.4) that the integrals on both sides of (2) exist. One may state (2) in more compact differential form as

$$(3) \quad d\phi(g) = \phi'(g) dg$$

with the understanding that this is just shorthand for (2). Because (3) generalizes the familiar chain rule for differentiating the composition of two functions, we shall hereafter refer to (2) as the *Riemann–Stieltjes chain rule*.

Let X be a continuous stochastic process on some probability space, and further suppose that $X(\omega)$ is a VF function for all $\omega \in \Omega$. (One may express this state of affairs more succinctly by saying that X is a *continuous VF process*.) For each fixed ω , apply (2) with $X(\omega)$ in place of g and $f(t) = 1$ for

all t . The left side then reduces to $\phi(X(t)) - \phi(X(0))$ and we arrive at the sample path relationship

$$(4) \quad \phi(X(t)) = \phi(X(0)) + \int_0^t \phi'(X) dX, \quad t \geq 0.$$

To repeat, (4) is a statement of equality between *random variables*; in the usual way, we suppress the dependence of X on ω to simplify typography. It is the purpose of Ito's formula, on which we focus in Chapter 4, to develop an analog of (4) for certain continuous processes X that do *not* have VF sample paths.

§5. NOTATIONAL CONVENTIONS FOR INTEGRALS

Where we have written $\int f dg$ to denote the Riemann–Stieltjes integral of f with respect to g , some authors write $\int f(t) dg(t)$. Because the latter notation involves so many unnecessary symbols, we shall use it only to show special structure for f or g . For example, the expressions

$$(1) \quad \int_0^T f dt,$$

$$(2) \quad \int_0^T e^{-\lambda t} dg, \quad \text{and}$$

$$(3) \quad \int_0^T e^{-\lambda t} h dt$$

may be written to signify integrals over $[0, T]$ where: in (1) the integrand is f and the integrator is $g(t) = t$; in (2) the integrand is $f(t) = \exp(-\lambda t)$; and in (3) the integrand is $f(t) = \exp(-\lambda t)h(t)$ and the integrator is $g(t) = t$. Occasionally, to conform with the usual notation for Riemann integrals, expressions (1) and (3) may be written as

$$\int_0^T f(t) dt \quad \text{and} \quad \int_0^T e^{-\lambda t} h(t) dt.$$

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